CHAPTER 8

Integration

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SECTION A **Integrals**

By the end of this section you will be able to:

- ▶ understand an indefinite integral as the anti-derivative
- ▶ use formulae to determine indefinite integrals

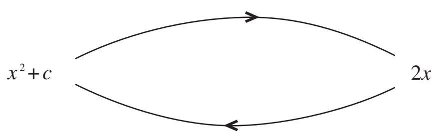
A1 **Integration – inverse of differentiation**

Integration is the **reverse** process of **differentiation**. It is sometimes called *anti-differentiation*. Consider a function $y = f(x)$. The anti-derivative of y is called the **indefinite integral** and is denoted by $\int y\,dx$. The integral notation, $\int y\,dx$, means integrate, $\int$, y with respect to x, dx. The function y is called the **integrand**.

For example, if we differentiate x^2 we obtain $2x$:

$$\frac{d}{dx}(x^2) = 2x$$

So does that mean the integral of $2x$ should be x^2?



No.

$$\int 2x\,dx = x^2 + C \text{ where } C \text{ is a constant}$$

Why do we need to add a constant C?

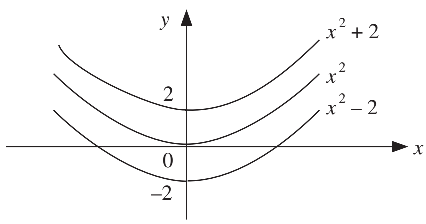
When we differentiate $x^2 + C$ we obtain $2x$ because differentiating a constant gives zero. It follows that

$$\int 2x\,dx = x^2 + C \text{ holds for any constant } C$$

There are an infinite number of different functions which are the integral of $2x$, that is why $\int 2x\,dx$ is called an indefinite integral.

Figure 1 shows some of the solutions to $\int 2x\,dx$.

Fig. 1



All these function, $x^2 + C$, are similar in nature apart from the constant C. This C is called the *constant of integration*. Of course other symbols can also represent the constant of integration. In general, if $y = f(x)$ and

$$\frac{dy}{dx} = f'(x) \text{ then integrating this gives } y = f(x) + C$$

For example, if $y = x^4$ then

$$\frac{dy}{dx} = 4x^3 \text{ and } \int 4x^3\,dx = x^4 + C$$

When we differentiate x^n we multiply by n and reduce the index of x by 1, thus

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

What is the reverse process of this?

We add 1 to the index of x and divide by the new index, $n + 1$, thus

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ (provided } n \neq -1)$$

Example 1

Determine

a $\int x^3 dx$ **b** $\int x^7 dx$ **c** $\int x^{1/2} dx$

Solution

To integrate, we add 1 to the index of x and divide by 'index plus 1'.

a $\int x^3 dx = \frac{x^{3+1}}{3+1} + C = \frac{x^4}{4} + C$

b Similarly we have

$$\int x^7 dx = \frac{x^{7+1}}{7+1} + C = \frac{x^8}{8} + C$$

c Even if the index of x is not a whole number, we still apply the same rule:

$$\int x^{1/2} dx = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C$$

$$= \frac{2x^{3/2}}{3} + C \quad \left[\text{Because } \frac{1}{3/2} = \frac{2}{3}\right]$$

Let's consider other functions. **What is the derivative $\frac{d}{du}[\sin(u)]$ equal to?**

$$\frac{d}{du}[\sin(u)] = \cos(u)$$

What is $\int \cos(u)\,du$ equal to?

Since integration is the reverse of differentiation we have

$$\int \cos(u)\,du = \sin(u) + C$$

? Hence the integral of cos is sin. **What is $\int e^u\,du$ equal to?**

Well $\frac{d}{du}(e^u) = e^u$, therefore

$$\int e^u du = e^u + C \quad \text{[Anti-derivative]}$$

Also from Chapter 6 on differentiation, the derivative of a natural logarithm, ln, is

$$\frac{d}{du}[\ln(u)] = \frac{1}{u}$$

? **What is $\int \frac{du}{u}$ equal to?**

$$\int \frac{du}{u} = \ln|u| + C$$

We place u with a modulus sign, | |, because the logarithmic function is only defined for the positive real numbers.

Following in the above manner we can establish an integration table of general engineering functions by reversing the differentiation process. There are many more functions in the integration table (Table 1) than there were in the differentiation table (Table 3 of Chapter 6) because integration is **more difficult** than differentiation. Don't be put off by the many complicated functions. We use the formulae in this table to find the integrals of various functions.

TABLE 1

	Add a constant (C) to all of these
8.1	$\int u^n du = \frac{u^{n+1}}{n+1}\ (n \neq -1)$
8.2	$\int \frac{du}{u} = \ln\|u\|$
8.3	$\int e^u du = e^u$
8.4	$\int a^u du = a^u/\ln(a)$
8.5	$\int \ln(u)du = u[\ln(u) - 1]$
8.6	$\int \log(u)du = u[\log(u) - \log(e)]$
8.7	$\int \sin(u)du = -\cos(u)$
8.8	$\int \cos(u)du = \sin(u)$

TABLE 1 *CONTINUED*

8.9	$\int \tan(u)\mathrm{d}u = \ln\lvert\sec(u)\rvert$
8.10	$\int \sec(u)\mathrm{d}u = \ln\lvert\sec(u) + \tan(u)\rvert$
8.11	$\int \mathrm{cosec}(u)\mathrm{d}u = \ln\lvert\mathrm{cosec}(u) - \cot(u)\rvert$
8.12	$\int \cot(u)\mathrm{d}u = \ln\lvert\sin(u)\rvert$
8.13	$\int \sin^{-1}(u)\mathrm{d}u = u\sin^{-1}(u) + \sqrt{1-u^2}$
8.14	$\int \cos^{-1}(u)\mathrm{d}u = u\cos^{-1}(u) - \sqrt{1-u^2}$
8.15	$\int \tan^{-1}(u)\mathrm{d}u = u\tan^{-1}(u) - \frac{1}{2}\ln(1+u^2)$
8.16	$\int \sinh(u)\mathrm{d}u = \cosh(u)$
8.17	$\int \cosh(u)\mathrm{d}u = \sinh(u)$
8.18	$\int \tanh(u)\mathrm{d}u = \ln[\cosh(u)]$
8.19	$\int \mathrm{sech}(u)\mathrm{d}u = \tan^{-1}[\sinh(u)]$
8.20	$\int \mathrm{cosech}(u)\mathrm{d}u = \ln\lvert\tanh(u/2)\rvert$
8.21	$\int \coth(u)\mathrm{d}u = \ln\lvert\sinh(u)\rvert$
8.22	$\int \sinh^{-1}(u)\mathrm{d}u = u\sinh^{-1}(u) - \sqrt{1+u^2}$
8.23	$\int \cosh^{-1}(u)\mathrm{d}u = u\cosh^{-1}(u) - \sqrt{u^2-1} \qquad (\cosh^{-1}(u) \geq 0)$
8.24	$\int \tanh^{-1}(u)\mathrm{d}u = u\tanh^{-1}(u) + \frac{1}{2}\ln(1-u^2)$
8.25	$\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)$
8.26	$\int \frac{\mathrm{d}u}{u^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$
8.27	$\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right)$
8.28	$\int \frac{\mathrm{d}u}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) = \ln\left\lvert u + \sqrt{u^2+a^2}\right\rvert$

TABLE 1 CONTINUED

8.29	$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) = \ln\left| u + \sqrt{u^2 - a^2} \right|$
8.30	$\int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$
8.31	$\int \sqrt{u^2 \pm a^2}\,du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|u + \sqrt{u^2 \pm a^2}\right|$
8.32	$\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right)$
8.33	$\int e^{au}\cos(bu)\,du = \frac{e^{au}}{a^2 + b^2}[a\cos(bu) + b\sin(bu)]$
8.34	$\int e^{au}\sin(bu)\,du = \frac{e^{au}}{a^2 + b^2}[a\sin(bu) - b\cos(bu)]$

Remember that u in this table is a dummy variable and can be replaced by any letter. For example, we could have replaced the u in this table by x.

Let p and q be functions of x, then we have

8.35 $$\int (p + q)\,dx = \int p\,dx + \int q\,dx$$

8.36 $$\int (p - q)\,dx = \int p\,dx - \int q\,dx$$

8.37 $$\int kp\,dx = k\int p\,dx \text{ where } k \text{ is a constant}$$

8.35 and 8.36 mean that we can take the integral of each function separately and then add or subtract the resulting integrals. 8.37 means that we can take a constant outside the integral and then multiply the result by the constant.

You can also use the differentiation table in reverse to find the integrals of functions not listed in Table 1 above.

Example 2

Determine

a $\int x^6\,dx$ **b** $\int (x^3 - 2x^2 + 5x)\,dx$ **c** $\int \frac{2}{\sqrt{x}}\,dx$

Solution

a $\int x^6\,dx \underbrace{=}_{\text{by 8.1 with } n=6} \frac{x^{6+1}}{6+1} + C = \frac{x^7}{7} + C$

8.1 $\int u^n\,du = \frac{u^{n+1}}{n+1} \quad (n \neq -1)$

Example 2 *continued*

b Splitting the integrand and taking out the constants we have

$$\int (x^3 - 2x^2 + 5x)dx = \int x^3 dx - 2\int x^2 dx + 5\int x\,dx$$

$$\underset{\text{by } 8.1}{=} \frac{x^{3+1}}{3+1} - 2\left[\frac{x^{2+1}}{2+1}\right] + 5\left[\frac{x^{1+1}}{1+1}\right] + C$$

$$= \frac{x^4}{4} - \frac{2x^3}{3} + \frac{5x^2}{2} + C$$

c Which formula should we use to find $\int \frac{2}{\sqrt{x}}dx$**?**

We need to rewrite $\frac{2}{\sqrt{x}}$: $\frac{2}{\sqrt{x}} = \frac{2}{x^{1/2}} = 2x^{-1/2}$

We have

$$\int \frac{2}{\sqrt{x}}dx = \int 2x^{-1/2}dx = 2\int x^{-1/2}dx \quad \text{[Taking out 2]}$$

$$= 2\left[\frac{x^{-\frac{1}{2}+1}}{{}^{-1}/_2 + 1}\right] + C$$

$$= \frac{2x^{\frac{1}{2}}}{1/2} + C = 4x^{1/2} + C \quad \left[\text{Because } \frac{2}{1/2} = 4\right]$$

Example 3 *thermodynamics*

Find **a** $\int PV^{1.3}dV$ **b** $\int \frac{P}{V}\,dV$

where P and V represent pressure and volume respectively. P is constant and V varies.

Solution

a $\int PV^{1.3}dV = P\int V^{1.3}dV$ [Taking out P]

$$= P\underbrace{\left[\frac{V^{1.3+1}}{1.3+1}\right]}_{\text{by } 8.1 \text{ with } n=1.3} + C = \frac{PV^{2.3}}{2.3} + C$$

8.1 $\int x^n dx = \frac{x^{n+1}}{n+1}$

Example 3 *continued*

b How do we determine $\int \frac{P}{V}\,dV$?

$$\int \frac{P}{V}\,dV = P\int \frac{dV}{V} = P\underbrace{\ln(V)}_{\text{by 8.2}} + C$$

We can write $\ln|V| = \ln(V)$ because V is volume and therefore is positive.

Example 4

Determine

a $\int \sin(\omega t)d(\omega t)$ **b** $\int [3e^t + 2\cos(t)]dt$ **c** $\int \left(\frac{1}{x} - x^{-3} + 5x^{3/5}\right)dx$

Solution

a From Table 1, using $\int \sin(u)du = -\cos(u) + C$ with $u = \omega t$ we have

$$\int \sin(\omega t)d(\omega t) = -\cos(\omega t) + C$$

b We can integrate each part and take the constants out, thus

$$\int [3e^t + 2\cos(t)]dt = 3\int e^t dt + 2\int \cos(t)dt$$
$$= \underbrace{3e^t}_{\text{by 8.3}} + \underbrace{2\sin(t)}_{\text{by 8.8}} + C$$

c Again splitting the integrand we have

$$\int \left(\frac{1}{x} - x^{-3} + 5x^{3/5}\right)dx = \int \frac{dx}{x} - \int x^{-3}dx + 5\int x^{3/5}dx$$
$$= \underbrace{\ln|x|}_{\text{by 8.2}} - \underbrace{\frac{x^{-3+1}}{-3+1}}_{\text{by 8.1}} + 5\underbrace{\frac{x^{3/5+1}}{3/5+1}}_{\text{by 8.1}} + C$$
$$= \ln|x| - \frac{x^{-2}}{-2} + 5\frac{x^{8/5}}{8/5} + C \quad \text{[Simplifying]}$$
$$= \ln|x| + \frac{x^{-2}}{2} + 25\frac{x^{8/5}}{8} + C \quad \left[\text{Because } \frac{5}{8/5} = \frac{25}{8}\right]$$

8.1 $\int x^n dx = \frac{x^{n+1}}{n+1}$ 8.2 $\int du/u = \ln|u|$ 8.3 $\int e^t dt = e^t$

8.8 $\int \cos(t)dt = \sin(t)$

SUMMARY

The **indefinite integral** of $y = f(x)$ is defined as the **anti-derivative** of y and is denoted by $\int y\,dx$.

Let p and q be functions of x then

8.35 $$\int (p + q)dx = \int p dx + \int q dx$$

8.36 $$\int (p - q)dx = \int p dx - \int q dx$$

8.37 $$\int kp dx = k \int p dx \text{ where } k \text{ is a constant}$$

Exercise **8(a)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Determine the following integrals:

a $\int x dx$ **b** $\int u^2 du$

c $\int z^3 dz$ **d** $\int t^{1/2} dt$

e $\int (\omega t) d(\omega t)$ **f** $\int 3 dt$

g $\int t^{-2} dt$ **h** $\int x^{-1/2} dx$

2 Find

a $\int \cos(\omega t) d(\omega t)$ **b** $\int 10^x dx$

c $\int \sinh(t) dt$ **d** $\int \sec(x) dx$

3 [*thermodynamics*] Find

a $\int PV^{1.35} dV$

b $\int PV^{1.61} dV$

(P is constant and V varies.)

4 Find

a $\int (x^2 + 2x) dx$

b $\int \left(\frac{1}{\sqrt{x}} + \tan(x)\right) dx$

c $\int (\sec^2(x) - 5e^x + 1) dx$

Hint: Use the differentiation table in reverse to find the integral of $\sec^2(x)$.

d $\int (\sin(x) + 2\sqrt{x} - 1) dx$

5 **i** Show that

$$\frac{d}{dx}\left(\frac{3 - 4x}{1 + x^2}\right) = \frac{4x^2 - 6x - 4}{(1 + x^2)^2}$$

ii Determine $\int \frac{4x^2 - 6x - 4}{(1 + x^2)^2} dx$

SECTION B Integration by substitution

By the end of this section you will be able to:

- ▶ use substitution to find indefinite integrals
- ▶ apply integration to engineering problems

B1 Using substitution

Example 5

Find $\int \cos(2x + 5)\mathrm{d}x$

Solution

What formula of Table 1 can we use to determine

$$\int \cos(2x + 5)\mathrm{d}x?$$

Using 8.8, $\int \cos(u)\mathrm{d}u = \sin(u) + C$ with $u = 2x + 5$, we have

$$* \qquad \int \cos(2x + 5)\mathrm{d}x = \int \cos(u)\mathrm{d}x = ?$$

We **cannot** use 8.8 to determine * because we have a dx in place of a du. **What can we substitute for dx?**

We know $u = 2x + 5$ and we can differentiate this function to obtain

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \text{ which gives } \mathrm{d}u = 2\mathrm{d}x$$

because $\mathrm{d}u = \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$, where $\frac{\mathrm{d}u}{\mathrm{d}x}$ is the 'differential coefficient' and du is the 'differential'.

Therefore we have $\mathrm{d}x = \frac{\mathrm{d}u}{2}$. Putting this into *:

$$\int \cos(u)\frac{\mathrm{d}u}{2} = \frac{1}{2}\int \cos(u)\mathrm{d}u = \frac{1}{2}\underbrace{\sin(u)}_{\text{by } 8.8} + C$$

It follows that

$$\int \cos(2x + 5)\mathrm{d}x = \frac{1}{2}\sin(u) + C = \frac{1}{2}\sin\underbrace{(2x + 5)}_{\substack{\text{substituting} \\ u = 2x+5}} + C$$

Example 6

Determine $\int \sin(2x)\mathrm{d}x$

Solution

We can use 8.7, $\int \sin(u)\mathrm{d}u = -\cos(u) + C$. Let $u = 2x$ then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \qquad \mathrm{d}x = \frac{\mathrm{d}u}{2}$$

Substituting $u = 2x$ and $\mathrm{d}x = \frac{\mathrm{d}u}{2}$ gives

$$\begin{aligned}\int \sin(2x)\mathrm{d}x &= \int \sin(u)\frac{\mathrm{d}u}{2}\\ &= \frac{1}{2}\int \sin(u)\mathrm{d}u \quad \left[\text{Taking out } \frac{1}{2}\right]\\ &= \frac{1}{2}[-\cos(u)] + C = -\frac{\cos(2x)}{2} + C\end{aligned}$$

In general we have (k and m are constants):

8.38 $$\int \cos(kx + m)\mathrm{d}x = \frac{\sin(kx + m)}{k} + C$$

8.39 $$\int \sin(kx + m)\mathrm{d}x = -\frac{\cos(kx + m)}{k} + C$$

8.40 $$\int \sec^2(kx + m)\mathrm{d}x = \frac{\tan(kx + m)}{k} + C$$

Example 7

Obtain $\int e^{7x+3}\mathrm{d}x$

Solution

? **Which formula can we use to find $\int e^{7x+3}\mathrm{d}x$?**

We use 8.3, $\int e^u\mathrm{d}u = e^u + C$. Let $u = 7x + 3$ then we have

$$\int e^{7x+3}\mathrm{d}x = \int e^u\mathrm{d}x$$

? **We need to replace the dx. How?**

Differentiating $u = 7x + 3$ gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 7 \qquad \mathrm{d}x = \frac{\mathrm{d}u}{7}$$

Substituting these: $\int e^{7x+3}\mathrm{d}x = \int e^u\frac{\mathrm{d}u}{7} = \frac{1}{7}\int e^u\mathrm{d}u = \frac{1}{7}e^u + C = \frac{1}{7}e^{7x+3} + C$

In general

8.41 $$\int e^{kx+m}\mathrm{d}x = \frac{e^{kx+m}}{k} + C$$

B2 **Engineering application**

Let's investigate an engineering example. The next example is particularly challenging. It is sophisticated compared to the above examples. Follow it through carefully.

Example 8 *structures*

The equation, y, of a catenary formed by a cable under its own weight is given by

$$y = \int \sinh\left(\frac{wx}{T}\right)\mathrm{d}x$$

where x is horizontal distance, T is horizontal tension and w is weight per unit length of cable. (T and w are constants.) Given that when $x = 0,\ y = 0$, show that

$$y = \frac{T}{w}\left[\cosh\left(\frac{wx}{T}\right) - 1\right]$$

Solution

We use **8.16**, $\int \sinh(u)\mathrm{d}u = \cosh(u) + C$ with $u = \frac{wx}{T}$. Differentiating:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{w}{T} \qquad \mathrm{d}x = \frac{T}{w}\mathrm{d}u$$

Putting these into the given integral:

$$y = \int \sinh\left(\frac{wx}{T}\right)\mathrm{d}x = \frac{T}{w}\int \sinh(u)\mathrm{d}u$$

$$= \frac{T}{w}\cosh(u) + C = \frac{T}{w}\cosh\left(\frac{wx}{T}\right) + C$$

Substituting $x = 0,\ y = 0$ into $y = \frac{T}{w}\cosh\left(\frac{wx}{T}\right) + C$ and using $\cosh(0) = 1$:

$$0 = \frac{T}{w} + C \text{ gives } C = -\frac{T}{w}$$

Hence

$$y = \frac{T}{w}\cosh\left(\frac{wx}{T}\right) - \frac{T}{w} = \frac{T}{w}\left[\cosh\left(\frac{wx}{T}\right) - 1\right] \quad \text{[Factorizing]}$$

B3 An important integral

Example 9

Obtain $\int \frac{3x^2}{x^3 - 5}\, dx$

Solution

What can we use to find $\int \frac{3x^2}{x^3 - 5}\, dx$?

We can try using $\int \frac{du}{u}$. Let $u = x^3 - 5$. We need to replace dx. **How can we find dx?**

Differentiating $u = x^3 - 5$ with respect to x gives

$$\frac{du}{dx} = 3x^2 \qquad dx = \frac{du}{3x^2}$$

$$\int \frac{3x^2}{x^3 - 5}\, dx = \int \left(\frac{3x^2}{u}\right)\frac{du}{3x^2} = \int \frac{du}{u} \quad \text{[Cancelling } 3x^2\text{]}$$

$$\underbrace{=}_{\text{by } 8.2} \ln|u| + C = \ln|x^3 - 5| + C$$

Take another look at $\int \frac{3x^2}{x^3 - 5}\, dx = \ln|x^3 - 5| + C$.

What do you notice about this integrand, $\frac{3x^2}{x^3 - 5}$?

Similarly we have

$$\int \frac{2x}{x^2 - 1}\, dx = \ln|x^2 - 1| + C$$

$$\int \frac{2x - 2}{x^2 - 2x}\, dx = \ln|x^2 - 2x| + C$$

$$\int \frac{5x^4 - 6x}{x^5 - 3x^2}\, dx = \ln|x^5 - 3x^2| + C$$

What do you notice about all these integrands and associated solutions to the integral?

In each case the numerator is the derivative of the denominator. That is, if you differentiate the denominator you get the numerator. In **Example 9** notice that the derivative, $3x^2$, of the denominator cancels out with the numerator and that is why we can use the $\int \frac{du}{u}$ formula. Also observe that the solution of these integrals is the natural log, ln, of the denominator.

8.2 $\int du/u = \ln|u|$

Example 10

Show that

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Solution

Let $u = f(x)$, then differentiating yields

$$\frac{du}{dx} = f'(x) \text{ therefore } dx = \frac{du}{f'(x)}$$

Substituting $u = f(x)$ and $dx = \dfrac{du}{f'(x)}$ into the given integral:

$$\begin{aligned}\int \frac{f'(x)}{f(x)}\,dx &= \int \left(\frac{f'(x)}{u}\right)\frac{du}{f'(x)}\\ &= \int \frac{du}{u} \quad \text{[Cancelling } f'(x)\text{]}\\ &= \ln|u| + C = \ln|f(x)| + C \quad \text{[Replacing } u = f(x)\text{]}\end{aligned}$$

In general if $f(x)$ is a function of x and its derivative is $f'(x)$ then

8.42 $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

8.42 is a **very important** integral because it can be used to integrate a wide range of functions such as

$$\int \frac{2x}{x^2+1}\,dx = \ln\left|x^2+1\right| + C \qquad \left[\text{Because } \frac{d}{dx}[x^2+1] = 2x\right]$$

$$\int \frac{\cos(x)}{\sin(x)}\,dx = \ln\left|\sin(x)\right| + C \qquad \left[\text{Because } \frac{d}{dx}[\sin(x)] = \cos(x)\right]$$

$$\int \frac{2}{2x+1}\,dx = \ln\left|2x+1\right| + C \qquad \left[\text{Because } \frac{d}{dx}[2x+1] = 2\right]$$

Say we had 3 in place of 2 in the above example, that is $\int \frac{3}{2x+1}\,dx$.

? Then how do we integrate this?

Differentiating $2x + 1$ with respect to x gives 2, so we have to rewrite the integrand as follows:

$$\frac{3}{2x+1} = \frac{3}{2}\,\frac{2}{2x+1}$$

Thus

$$\begin{aligned}\int \frac{3}{2x+1}\,dx &= \frac{3}{2}\int \frac{2}{2x+1}\,dx\\ &= \frac{3}{2}\ln\left|2x+1\right| + C \qquad \text{[Integrating]}\end{aligned}$$

This is normal procedure in these types of examples. Normally you have to rewrite the integrand so that you can apply formula 8.42. Similarly we have

$$\int \frac{5}{3x+2}\,dx = \frac{5}{3}\ln|3x+2| + C$$

$$\int \frac{x+1}{x^2+2x+5}\,dx = \frac{1}{2}\ln|x^2+2x+5| + C$$

The last result might be more difficult to follow. Differentiating $x^2 + 2x + 5$ with respect to x gives $2x + 2$ and so rewriting the integrand we have

$$\frac{x+1}{x^2+2x+5} = \frac{1}{2}\,\frac{2x+2}{x^2+2x+5}$$

Thus

$$\int \frac{x+1}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+5}\,dx$$
$$= \frac{1}{2}\ln|x^2+2x+5| + C$$

Example 11

Determine $\int \frac{t^2}{t^3-1}\,dt$

Solution

Differentiating $t^3 - 1$ gives us $3t^2$ and we are missing the 3 in the numerator. But it is close so let's try using the $\int \frac{du}{u}$ formula. Let $u = t^3 - 1$ and differentiating:

$$\frac{du}{dt} = 3t^2 \text{ which gives } dt = \frac{du}{3t^2}$$

Substituting these gives

$$\int \frac{t^2}{t^3-1}\,dt = \int \left(\frac{t^2}{u}\right)\frac{du}{3t^2} \underbrace{=}_{\text{cancelling } t^2} \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\underbrace{|t^3-1|}_{\text{replacing } u = t^3-1} + C$$

If you know that the derivative of the denominator is 3 times the numerator then take out 1/3 and integrate the remaining function of the form $\frac{f'(x)}{f(x)}$.

SUMMARY

An important integral well worth remembering is

8.42 $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Exercise **8(b)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Obtain **a** $\int \sin(7x+1)\mathrm{d}x$

b $\int \cos(7x+1)\mathrm{d}x$

2 Find **a** $\int \cos(\omega t)\mathrm{d}t$ **b** $\int \cos(\omega t+\theta)\mathrm{d}t$

3 Obtain **a** $\int \sin(\omega t)\mathrm{d}t$ **b** $\int \sin(\omega t)\,\mathrm{d}(\omega t)$

4 Determine **a** $\int \frac{2x}{x^2-1}\mathrm{d}x$

b $\int \frac{3x^2-6x}{x^3-3x^2+1}\mathrm{d}x$

5 Show that $\int \cot(x)\mathrm{d}x = \ln|\sin(x)| + C$

6 Show that

a $\int \tanh(x)\,\mathrm{d}x = \ln|\cosh(x)| + C$

b $\int \coth(x)\,\mathrm{d}x = \ln|\sinh(x)| + C$

7 Find **a** $\int \frac{\mathrm{d}t}{7t-1}$ **b** $\int \frac{t^3}{t^4-1}\,\mathrm{d}t$

c $\int \frac{t^2}{5-t^3}\,\mathrm{d}t$

8 Determine **a** $\int e^{11x+5}\,\mathrm{d}x$ **b** $\int e^{-2x+1000}\,\mathrm{d}x$

Questions 9 to 12 belong to the field of [*mechanics*].

9 The vertical velocity component, v, of a projectile is defined as

$$v = \int (-g)\mathrm{d}t$$

where g is the constant acceleration due to gravity. Given that at $t = 0$, $v = v_0$, show that

$$v = v_0 - gt$$

10 The velocity, v, of a projectile is given by

$$v = \int (-g)\mathrm{d}t$$

Given that when $t = 0$, $v = u$, find an expression for v in terms of t.

11 The position, s, of a particle moving along a straight line is given by

$$s = \int 10(30t+1)^{-1/2}\mathrm{d}t$$

Given that when $t = 0$, $s = \frac{2}{3}$, show that

$$s = \frac{2}{3}(30t+1)^{1/2}$$

12 The velocity, v, of a particle is given by

$$v = \int (-6t)\mathrm{d}t$$

For the initial condition that when $t = 0$, $v = 48$ m/s, find

i an expression for the velocity, v

ii the time taken, t, for the velocity to become $v = 0$

13 [*fluid mechanics*] The pressure, P, and density, ρ, of an airstream in an isentropic flow are related by

$$\frac{P}{\rho^{\gamma}} = k$$

where k is a constant and γ is the specific heat constant. Find $\int \frac{\mathrm{d}P}{\rho}$.

SECTION C Definite integrals

By the end of this section you will be able to:

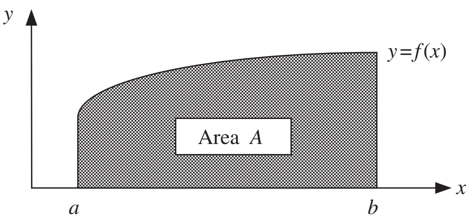
- ▶ understand the definite integral
- ▶ evaluate definite integrals for engineering examples

C1 Definite integrals

Say we want to find the area A under the curve $y = f(x)$ between a and b of Fig. 2.

How can we obtain this area?

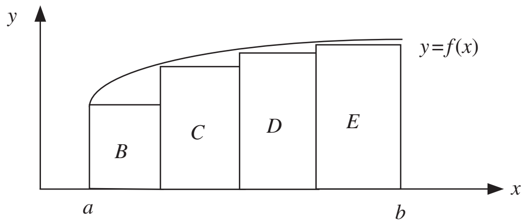
Fig. 2



It is not a regular shape like a circle or a rectangle so we cannot use any of the established formulae such as πr^2 or length × height.

We can cut the area A into smaller rectangular blocks as shown in Fig. 3.

Fig. 3



So does the area A equal $B + C + D + E$?

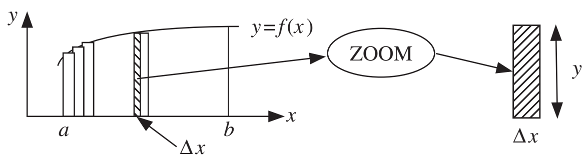
No, because $B + C + D + E$ does **not** cover all the area A.

Some of the area under the curve is **not** covered by the rectangular blocks.

How can we obtain a more accurate answer for the area A?

The area A can be chopped into even smaller pieces of width Δx as shown in Fig. 4.

Fig. 4



If the width Δx is small enough then we can obtain an approximation for the entire area A.

Consider the shaded area of Fig. 4. **How can we find this area?**

The shaded area is a rectangle of width Δx and height y:

$$\text{Rectangle area} = y\Delta x$$

How can we obtain an approximation to the original area A under the curve?

By adding together all the rectangles of height y and width Δx.

Observe that the height y changes as you move along the curve. Hence the area A is approximately the sum from a to b of $y\Delta x$ and can be written as

$$\text{Area } A \simeq S_a^b y\Delta x$$

where S_a^b denotes the sum from a to b.

The approximation becomes closer and closer to the exact area A as Δx gets smaller and smaller. To obtain the exact area A we need Δx to be very close to zero, $\Delta x \to 0$. We have

$$\text{Area } A = \lim_{\Delta x \to 0} S_a{}^b y \Delta x = S_a{}^b y \, \mathrm{d}x$$

where dx has replaced Δx as $\Delta x \to 0$. [In the notation, over the years, the letter S was replaced by an elongated S, $\int$.]

$$\text{Area } A = \int_a^b y \, \mathrm{d}x$$

Some of this notation might seem baffling but after a few examples it will become pretty straightforward. The notation was introduced by the German mathematician Liebniz who thought of obtaining the area under the curve by summing areas of rectangles. Let's define some of the terms.

$\int_a^b y \, \mathrm{d}x$ is called the **definite integral** of y. The process of determining the area is called integration and $x = a$, $x = b$ are called the **limits of integration.** Remember that y is a function of x, $y = f(x)$.

We can witness how the area under the curve can be determined by a definite integral by displaying graphs in a computer algebra system such as MAPLE.

If you don't have access to MAPLE or a similar package such as MATLAB, MATHEMETICA, DERIVE etc., read through the example carefully and see if you understand the results.

Example 12

Plot the graph of $y = x^2$ from 0 to 3.

i Apply the leftbox command in MAPLE to draw 5 boxes on the graph of $y = x^2$ and find the total area of these boxes by using the command leftsum.

ii Repeat **i** for 10 boxes.

iii Repeat **i** for 50 boxes.

iv Repeat **i** for 100 boxes.

v Evaluate $\int_0^3 x^2 \, \mathrm{d}x$ by using the evalf (int(x^2, x = 0..3)); command

What do you notice about your graphs and their corresponding total area evaluation?

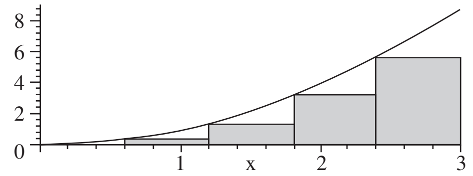
Solution

```
> with (student):
```

Warning, new definition for D

Example 12 *continued*

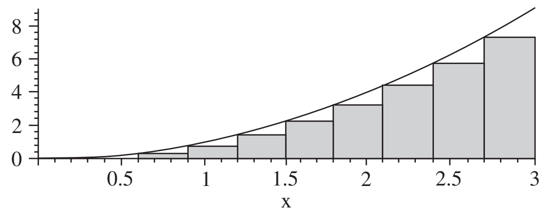
```
> leftbox (x^2,x=0..3,5);
```




```
> evalf (leftsum(x^2,x=0..3,5));
```

6.480000000

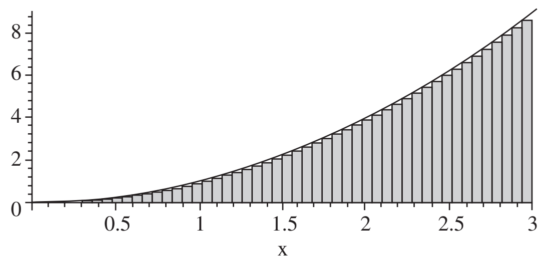
```
> leftbox (x^2,x=0..3,10);
```




```
> evalf (leftsum(x^2,x=0..3,10));
```

7.695000000

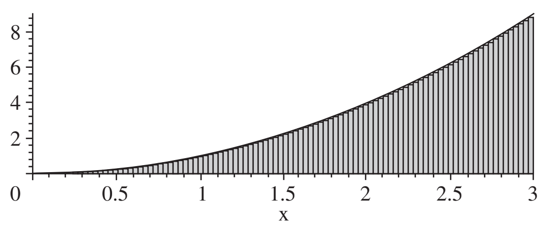
```
> leftbox (x^2,x=0..3,50);
```




```
> evalf (leftsum(x^2,x=0..3,50));
```

8.731800000

```
> leftbox (x^2,x=0..3,100);
```




```
> evalf (leftsum(x^2,x=0..3,100));
```

8.865450000

```
> evalf (int(x^2,x=0..3));
```

9.

Example 12 *continued*

Notice as the number of boxes increases, more of the area under the curve becomes covered. Using more boxes in MAPLE and summing the corresponding area (leftsum) gives the results shown in Table 2.

TABLE 2

Number of boxes	100	200	500	1000	100 000
Area (5 *d.p.*)	8.86545	8.93261	8.97302	8.98650	8.99987

If we consider more and more boxes, the limiting value of the sum will be 9. Thus the total area under the curve $y = x^2$ between 0 to 3 is 9. We have

$$\int_0^3 x^2\,dx = 9$$

As the number of boxes increases, the area gets closer and closer to the integration of x^2 from 0 to 3.

C2 **Engineering applications of the definite integral**

We defined the area under the curve $y = f(x)$, between $x = a$ to $x = b$ (Fig. 5) as

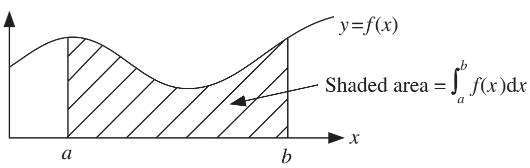
8.43 $$\int_a^b f(x)\,dx \left[= \int_{x=a}^{x=b} f(x)\,dx \right]$$

8.43 is called the definite integral.

If $\int f(x)\,dx = F(x)$ then

8.44 $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Fig. 5



In other words

$$\int_a^b f(x)\,dx = [\text{the value of the integral at } x = b] - [\text{the value of the integral at } x = a]$$

Example 13

Evaluate $\int_2^4 x^2 dx$.

Solution

We first integrate x^2 with respect to x. Then we evaluate the integral at $x = 4$ and $x = 2$ and subtract the two values. Notice how the 4 and 2 are transferred to the top right and bottom right of the square brackets after the integration is carried out.

Example 13 *continued*

$$\int_2^4 x^2 dx = \left[\frac{x^3}{3} + C\right]_2^4 \qquad \text{[Integrating by 8.1]}$$

$$= \underbrace{\left(\frac{4^3}{3} + C\right)}_{\text{substituting } x = 4} - \underbrace{\left(\frac{2^3}{3} + C\right)}_{\text{substituting } x = 2}$$

$$= \frac{64}{3} + C - \frac{8}{3} - C$$

$$\int_2^4 x^2 dx = \frac{56}{3}$$

? Note that the definite integral $\int_2^4 x^2 dx$ is a number, $\frac{56}{3}$, and does not contain x. **Moreover there is *no* constant of integration (C). Why?**

Because the C's cancel each other out. When there is a definite integral you do not need to include the constant C. Let's examine some engineering examples.

Example 14 *mechanics*

The displacement, s, of an object is given by

$$s = \int_0^3 (t^4 + t) dt$$

Evaluate s.

Solution

We integrate and then substitute $t = 3$ and $t = 0$:

$$s = \int_0^3 (t^4 + t) dt$$

$$= \left[\frac{t^5}{5} + \frac{t^2}{2}\right]_0^3 \qquad \text{[Integrating by 8.1]}$$

$$= \underbrace{\left(\frac{3^5}{5} + \frac{3^2}{2}\right)}_{\text{substituting } t = 3} - \underbrace{\left(\frac{0^5}{5} + \frac{0^2}{2}\right)}_{\text{substituting } t = 0} = 53.1$$

$$s = 53.1 \text{ m}$$

8.1 $\int u^n du = \frac{u^{n+1}}{n + 1}$

Example 15 *thermodynamics*

The change in specific enthalpy, Δh, in J/kg, is given by

$$\Delta h = \int_{150}^{300} \big(2.1 + (7 \times 10^{-3})T\big)\mathrm{d}T$$

Evaluate Δh.

Solution

Our limits of integration are $T = 300$ and $T = 150$:

$$\Delta h = \int_{150}^{300} \big(2.1 + (7 \times 10^{-3})T\big)\mathrm{d}T$$

$$= \left[2.1T + \frac{(7 \times 10^{-3})T^2}{2}\right]_{150}^{300} \qquad \text{[Integrating by 8.1]}$$

$$= \underbrace{\left((2.1 \times 300) + \frac{(7 \times 10^{-3})300^2}{2}\right)}_{\text{substituting } T = 300} - \underbrace{\left((2.1 \times 150) + \frac{(7 \times 10^{-3})150^2}{2}\right)}_{\text{substituting } T = 150}$$

$$= 945 - 393.75 = 551.25$$

$$\Delta h = 551.25 \text{ J/kg}$$

Sometimes in a definite integral we substitute a letter or a symbol rather than a number in the limits of integration. This results in an expression.

Example 16 *mechanics*

The moment of inertia, I, of a rod of length $2r$ is given by

$$I = \int_{-r}^{r} \frac{mx^2}{2r}\,\mathrm{d}x$$

where m is the mass of the rod and x is the distance from the axis. Show that $I = \frac{mr^2}{3}$. (The mass, m, and length, r, are constant.)

8.1 $\int u^n \mathrm{d}u = u^{n+1}/n + 1$

Example 16 *continued*

Solution

Taking out the $m/2r$ gives

$$I = \frac{m}{2r}\int_{-r}^{r} x^2 dx = \frac{m}{2r}\left[\frac{x^3}{3}\right]_{x=-r}^{x=r} = \frac{m}{2r}\left(\frac{r^3 - (-r)^3}{3}\right) = \frac{m}{2r}\left(\frac{2r^3}{3}\right) = \frac{mr^2}{3} \quad \text{[Cancelling]}$$

The next example is difficult because it involves trigonometric identities, integration, substitution, unfamiliar symbols etc. It is a lot more complicated than previous examples.

Example 17 *electrical principles*

The power, P, in a circuit is given by

$$P = \frac{\omega R}{2\pi}\int_0^{2\pi/\omega} (i^2) dt$$

where $i = I\sin(\omega t)$, R is resistance and ω is angular frequency. Show that

$$P = \frac{I^2 R}{2}$$

Solution

We have $i = I\sin(\omega t)$, therefore $i^2 = [I\sin(\omega t)]^2 = I^2\sin^2(\omega t)$. Substituting $i^2 = I^2\sin^2(\omega t)$ into $P = \frac{\omega R}{2\pi}\int_0^{2\pi/\omega} (i^2) dt$ gives

$$P = \frac{\omega R}{2\pi}\int_0^{2\pi/\omega} I^2\sin^2(\omega t)\, dt = \frac{\omega R I^2}{2\pi}\int_0^{2\pi/\omega} \sin^2(\omega t) dt \quad \text{[Taking out } I^2\text{]}$$

The difficulty in this example is to find $\int_0^{2\pi/\omega} \sin^2(\omega t) dt$.

How do we integrate the $\sin^2$ term?

We need to avoid $\sin^2$ by finding another way of expressing it.

We use the trigonometric identity

4.68 $$\sin^2(x) = \frac{1}{2}\left[1 - \cos(2x)\right]$$

$$\sin^2(\omega t) = \frac{1}{2}\left[1 - \cos(2\omega t)\right]$$

Example 17 *continued*

Therefore we have

$$P = \frac{\omega R I^2}{2\pi}\int_0^{2\pi/\omega} \sin^2(\omega t)\,\mathrm{d}t = \frac{\omega R I^2}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2}\left[1 - \cos(2\omega t)\right]\mathrm{d}t$$

$$= \frac{\omega R I^2}{4\pi}\int_0^{2\pi/\omega}\left[1 - \cos(2\omega t)\right]\mathrm{d}t \quad \left[\text{Taking out } \frac{1}{2}\right]$$

$$= \frac{\omega R I^2}{4\pi}\left[t - \frac{\sin(2\omega t)}{2\omega}\right]_0^{2\pi/\omega} \qquad \left[\text{Integrating using } \int \cos(kt)\mathrm{d}t = \frac{\sin(kt)}{k}\right]$$

$$= \frac{\omega R I^2}{4\pi}\left\{\underbrace{\left(\frac{2\pi}{\omega} - \frac{\sin[2\omega\,(2\pi/\omega)]}{2\omega}\right)}_{\text{substituting } t = 2\pi/\omega} - \underbrace{\left[0 - \frac{\sin(0)}{2\omega}\right]}_{\text{substituting } t = 0}\right\}$$

$$= \frac{\omega R I^2}{4\pi}\left[\left(\frac{2\pi}{\omega} - \underbrace{\frac{\sin(4\pi)}{2\omega}}_{\substack{= 0 \text{ because} \\ \sin(4\pi) = 0}}\right) - 0\right]$$

$$= \frac{\omega R I^2}{4\pi}\left[\frac{2\pi}{\omega}\right] = \frac{I^2 R}{2} \qquad \text{[Cancelling]}$$

$$P = \frac{I^2 R}{2}$$

Example 18 *reliability engineering*

The failure density function, $f(t)$, for a class of electronic components is given by

$$f(t) = 0.2e^{-0.2t}$$

The hazard function, $h(t)$, is defined as

$$h(t) = \frac{f(t)}{1 - \int_0^t f(x)\mathrm{d}x}$$

Determine $h(t)$.

Example 18 *continued*

Solution

We are given $f(t)$, but what is $f(x)$ equal to?

Since $f(x)$ is a function of x, we replace the t with x, thus $f(x) = 0.2e^{-0.2x}$.

$$\int_0^t f(x)\mathrm{d}x = 0.2\int_0^t e^{-0.2x}\mathrm{d}x \quad \text{[Taking out 0.2]}$$

$$= 0.2\left[\frac{e^{-0.2x}}{-0.2}\right]_0^t \quad \left[\text{Integrating using} \int e^{kx}\mathrm{d}x = \frac{e^{kx}}{k}\right]$$

$$= -\left[e^{-0.2x}\right]_0^t \qquad \text{[Cancelling 0.2's]}$$

$$= -\Big(\underbrace{e^{-0.2t}}_{\text{substituting } x = t} - \underbrace{e^{-(0.2\times 0)}}_{\text{substituting } x = 0}\Big)$$

$$= -(e^{-0.2t} - 1) = -e^{-0.2t} + 1$$

$$\int_0^t f(x)\mathrm{d}x = 1 - e^{-0.2t}$$

Simplifying the denominator in $h(t)$ gives

$$1 - \int_0^t f(x)\mathrm{d}x = 1 - (1 - e^{-0.2t})$$

$$= 1 - 1 + e^{-0.2t}$$

$$= e^{-0.2t}$$

Replacing $1 - \int_0^t f(x)\mathrm{d}x$ with $e^{-0.2t}$ in the denominator of the hazard function, $h(t)$, gives

$$h(t) = \frac{0.2\, e^{-0.2t}}{e^{-0.2t}} = 0.2 \qquad \text{[Cancelling } e^{-0.2t}\text{]}$$

Remember the variable x in $\int_a^b f(x)\mathrm{d}x$ is called the **dummy variable** since x can be replaced by any letter because the value of the integral is the same. That is

$\int_a^b f(x)\mathrm{d}x = \int_a^b f(t)\mathrm{d}t = \int_a^b f(\xi)\mathrm{d}\xi = \cdots$ (A definite integral is a function of the limits **only**.)

SUMMARY

The **definite integral** is defined as **the area under the curve**, $y = f(x)$, between $x = a$ and $x = b$. Area $= \int_a^b y \, dx$.

If $\int_a^b f(x)dx = F(x)$ then

$$\int_a^b f(x)dx = F(b) - F(a) \quad (8.44)$$

There is no constant (C) of integration in the evaluation of a definite integral.

Exercise 8(c)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Plot the graph of $y = x^3$ from $x = 0$ to 1.

- **i** Use the leftbox command to draw four boxes on the graph of $y = x^3$. Evaluate the total area of the four boxes by using the leftsum command.
- **ii** Repeat **i** for 8 boxes.
- **iii** Repeat **i** for 16 boxes.
- **iv** Repeat **i** for 32 boxes.
- **v** Repeat **i** for 64 boxes.
- **vi** Repeat **i** for 128 boxes.
- **vii** By applying the command evalf(int($x^{3,}x$ = 0..1)), evaluate $\int_0^1 x^3 dx$. What do you notice about your results?

Questions 2 to 6, inclusive, belong to the field of [*mechanics*].

2 A particle moves along a horizontal line with displacement, s, given by

$$s = \int_0^2 (t^2 - 2t)dt$$

Determine s.

3 The work done, W, to stretch a spring from its natural length to an extension of 0.5 m is given by

$$W = \int_0^{0.5} 100x\,dx$$

Evaluate W.

4 The force, F, required to compress a spring is given by

$$F = 1000x + 50x^3$$

where x is the displacement from its unstretched length. The work done, W, to compress a spring by 0.3 m is given by

$$W = \int_0^{0.3} F\,dx$$

Determine W.

5 The distance, s, travelled by a train on a straight track in the first two seconds is given by

$$s = \int_0^2 20(1 - e^{-t})dt$$

Find s.

6 The distance, s, covered by a vehicle between the 5th and 6th second is given by

$$s = \int_5^6 10e^{0.4t}dt$$

Evaluate s.

Exercise **8(c) continued**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

7 [*structures*] A beam of length 6 m has a uniform distributed load, w, given by

$$w = 800 + \frac{1}{2}x^3$$

where x is the distance along the beam. The total load P (N), and the moment about the origin, R (N m), are given by

$$P = \int_0^6 w\,dx \text{ and } R = \int_0^6 (wx)\,dx$$

Determine P and R.

8 [*thermodynamics*] The specific heat, c, of steam is given by

$$c = \frac{3000 + T - 100}{1300}$$

The enthalpy change, Δh, is given by

$$\Delta h = \int_{150}^{500} 1.5c\,dT$$

Evaluate Δh.

9 [*mechanics*] The time taken, t (hours), for a vehicle to reach a speed of 120 km/h with an initial speed of 80 km/h is given by

$$t = \int_{80}^{120} \frac{dv}{600 - 3v}$$

where v is velocity (km/h). Determine t.

10 [*thermodynamics*] A gas in a cylinder obeys the law $PV^{1.25} = 1789$. The work done, W, is given by

$$W = \int_{0.01}^{0.1} P\,dV$$

Determine W.

Questions 11 to 15, inclusive, are in the field of [*mechanics*].

11 The moment of inertia, I, of a disc of radius r and mass m is given by

$$I = \int_0^r \left(\frac{2mx^3}{r^2}\right)dx$$

where x is the distance from an axis of rotation. Show that $I = \frac{mr^2}{2}$.

12 The moment of inertia, I, of an annulus of inner radius a, outer radius b and mass m is given by

$$I = \int_a^b \frac{2mx^3}{b^2 - a^2}\,dx$$

where x is the distance from the axis of rotation. Show that $I = \frac{1}{2}m(a^2 + b^2)$.

13 The moment of inertia, I, of a rod of mass m and length $2r$ is given by

$$I = \int_0^{2r} \frac{mx^2}{2r}\,dx$$

where x is the distance from the axis of rotation. Show that $I = \frac{4mr^2}{3}$.

14 The moment of inertia, I, of a rod of length l and mass m is given by

$$I = \int_0^l \frac{mx^2\sin^2(\theta)}{l}\,dx$$

where x is the distance along the rod and θ is the angle made between the rod and axis of rotation. Show that

$$I = \frac{ml^2\sin^2(\theta)}{3}$$

15 The moment of inertia, J, of a disc of radius r and mass m is defined as

$$J = \int_0^r \frac{2m}{r^2}x^3\,dx$$

Evaluate J.

16 [*fluid mechanics*] The momentum per unit time, M, for a fluid flow through a pipe of radius r is given by

$$M = k\int_0^r x^{2/7}(r - x)\,dx$$

where x is the distance from the pipe wall and k is a constant. Evaluate M.

Exercise 8(c) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

17 [*fluid mechanics*] The rate of flow, Q, of a fluid at a radius r in a pipe of radius R is given by

$$Q = \int_0^R (2\pi ur)\,dr$$

where u is the mean velocity, which is constant. Evaluate Q.

18 [*signal processing*] The average voltage, v_{avg}, of a waveform with period T is given by

$$v_{avg} = \frac{1}{T}\int_0^T V\cos(\omega t + \theta)dt$$

where V is the peak voltage, ω is the angular frequency and θ is the phase. Evaluate v_{avg}.

19 [*signal processing*] The mean, $\bar{x}$, of a signal over a period T is given by

$$\bar{x} = \frac{1}{T}\int_0^T a\cos(\omega t + \phi)dt$$

where a is the amplitude, ω is the angular frequency and ϕ is the phase. Determine $\bar{x}$.

Questions 20 to 26, inclusive, are in the field of [*electrical principles*].

20 The average value of current, I_{AV}, for a rectified alternating current is given by

$$I_{AV} = \frac{\omega}{\pi}\int_0^{\pi/\omega} i\,dt$$

where $i = I\sin(\omega t)$, ω is angular frequency, i is instantaneous current, I is peak current and t is time. Show that

$$I_{AV} = \frac{2}{\pi}I$$

21 The energy, W, stored in a capacitor of capacitance C charged with voltage V, is defined by

$$W = \int_0^V CV\,dV$$

Show that $W = \frac{1}{2}CV^2$.

22 A resistor shaped in the form of an annulus has an outer radius b and inner radius a and its resistance R is given by

$$R = \int_a^b \frac{\rho\,dr}{2\pi r}$$

where $a < r < b$ and ρ is the resistance per unit area. Evaluate R.

23 The voltage, v, across a capacitor is given by

$$v = \frac{1}{10 \times 10^{-6}}\int_0^t 100e^{-(5\times 10^3)t}dt$$

Evaluate v.

24 The magnetising force, F, of a circular coil of radius r and at a distance d from the centre is given by

$$F = \int_0^{2\pi} \frac{I\,r\sin(\beta)}{4\pi(d^2 + r^2)}\,d\phi \quad \left[\frac{I\,r\sin(\beta)}{4\pi(d^2 + r^2)} \text{ is a constant}\right]$$

where I is the current carried by the coil, ϕ and β are angles. Evaluate F.

25 The magnetising force, F, of an infinite wire is given by

$$F = \int_0^{\pi} \frac{I\sin(\theta)}{4\pi r}\,d\theta$$

where r is perpendicular distance, θ is angle and I is current. Evaluate F.

26 The potential difference, V, between two concentric spheres of radii a and b is given by

$$V = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2}\,dr \text{ where } a < r < b \ (r \neq 0)$$

ε_0 is the permittivity constant and Q is charge. Show that

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) \text{(where } a \neq 0, b \neq 0)$$

Exercise **8(c) continued**

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Questions 27 to 30, inclusive, are in the field of [*reliability engineering*].

27 The hazard function, $h(t)$, for a component is given by

$$h(t) = t^2 - 4t + 9 \quad (0 < t \leq 1)$$

The failure density function, $f(t)$, is defined as

$$f(t) = h(t)\exp\left[-\int_0^t h(x)\mathrm{d}x\right]$$

where exp is the exponential function. Determine $f(t)$.

28 The hazard function, $h(t)$, for a component is given by

$$h(t) = 3 - t \quad (0 < t \leq 2)$$

Determine $f(t)$ where $f(t)$ is as defined in question **27**.

29 The failure density function, $f(t)$, for a set of components is given by

$$f(t) = 0.02(10 - t) \quad (0 < t \leq 10)$$

The reliability function, $R(t)$, is defined as

$$R(t) = 1 - \int_0^t f(x)\mathrm{d}x$$

and the hazard function, $h(t)$, is defined as

$$h(t) = \frac{f(t)}{R(t)}$$

Determine $R(t)$ and $h(t)$.

30 The hazard function, $h(t)$, is given by

$$h(t) = (2 \times 10^{-3})t^{-1/2}$$

The reliability function, $R(t)$, is defined as

$$R(t) = \exp\left[-\int_0^t h(x)\mathrm{d}x\right]$$

Find $R(t)$.

Questions 31 to 34, inclusive, are in the field of [*thermodynamics*].

31 The work done, W, by a gas on a piston is given by

$$W = \int_{v_1}^{v_2} \frac{C}{V}\,\mathrm{d}V$$

where C is a constant and V is volume. Show that $W = C\ln\left(\frac{v_2}{v_1}\right)$.

32 The work done, W, by a gas on a piston is given by

$$W = \int_{p_1}^{p_2} CP^{-1/n}\mathrm{d}P$$

where P is pressure and C is constant. Evaluate W.

33 A gas expands according to the law

$$P_1V_1^k = P_2V_2^k = C$$

where P is pressure, V is volume, k and C are constants. The work done, W, by the gas is given by

$$W = \int_{V_1}^{V_2} CV^{-k}\mathrm{d}V$$

Show that $W = \dfrac{P_1V_1 - P_2V_2}{k - 1}$.

34 A gas obeys the law

$$PV^{1.5} = C \quad \text{(constant)}$$

If the work done, $W = \int_{V_1}^{V_2} P\,\mathrm{d}V$ then show that

$$W = 2C\left[\frac{1}{\sqrt{V_1}} - \frac{1}{\sqrt{V_2}}\right]$$

35 [*aerodynamics*] The drag coefficient, C_F, is defined as

$$C_F = k\int_0^1 \left(\frac{x}{L}\right)^{-\frac{1}{2}}\mathrm{d}\left(\frac{x}{L}\right)$$

where k is a constant, L and x are lengths. Show that $C_F = 2k$.

SECTION D **Integration by parts**

By the end of this section you will be able to:

- understand the integration by parts formula
- apply the integration by parts formula
- apply the formula to engineering applications

This section might seem like a huge leap in sophistication from previous sections. The difficulty is in understanding the notation of the integration by parts formula.

D1 **Integration by parts formula**

The product rule for differentiation is given by

6.31 $$\frac{d}{dx}(uv) = u'v + uv'$$

where u and v are functions of x. The dash notation, $'$, represents derivatives. Integrating both sides of 6.31:

$$uv = \int u'v\,dx + \int uv'dx$$

Rearranging yields

8.45 $$\int uv'dx = uv - \int u'v\,dx$$

8.45 is called the **integration by parts** formula. It is used to integrate **some** product of functions. If you have never seen this formula you might be confused with the symbols and notation. The v on the Right-Hand Side is the integral of v', that is

$$v = \int v'dx$$

In general the verbal form for integration by parts formula is 'differentiate one (u) and integrate the other, (v')'.

Example 19

Find $\int xe^x dx$.

Solution

Let $u = x$ and $v' = e^x$. Then to use the integration by parts formula, 8.45, we need to differentiate u and integrate v':

$$u' = 1 \qquad v = \int e^x dx = e^x$$

Example 19 *continued*

Don't worry about the constant (C) of integration, we add it on at the end. Substituting these into 8.45:

$$\int xe^x \mathrm{d}x = uv - \int vu' \mathrm{d}x$$

$$= xe^x - \int (e^x)(1)\mathrm{d}x$$

$$= xe^x - \int e^x \mathrm{d}x$$

We have $\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x$. Notice that we still need to integrate e^x, the last term on the Right-Hand Side. Hence

$$\int xe^x \mathrm{d}x = xe^x - \underbrace{e^x}_{=\int e^x \mathrm{d}x} + C$$

$$= e^x(x-1) + C \qquad \text{[Taking out } e^x\text{]}$$

Observe that in the application of 8.45, we still have an integral that we need to determine, $\int vu' \mathrm{d}x$.

? **What is $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^x(x-1) + C\right]$?**

By inspecting **Example 19** we have

$$\frac{\mathrm{d}}{\mathrm{d}x}[e^x(x-1) + C] = xe^x$$

because when we integrate xe^x we obtain $e^x(x-1) + C$. Remember that differentiation is the reverse process of integration.

? **Why did we choose $u = x$ and $v' = e^x$ in Example 19?**

If you selected u and v' the other way round, that is $u = e^x$ and $v' = x$ then the integration of xe^x becomes very complicated. Applying 8.45 gives

$$\int xe^x \mathrm{d}x = e^x \frac{x^2}{2} - \int \frac{x^2}{2} e^x \mathrm{d}x$$

The integral on the Right-Hand Side is more complicated then the integral of the given function, xe^x.

8.45 $\int uv' \mathrm{d}x = uv - \int vu' \mathrm{d}x$

In general, how do we know what to take u and v' to be?

From previous experience we can establish some sort of ranking. Take u to be the function in the following order:

1. $\ln(x)$
2. x^n
3. e^{kx} where k is a constant

So if we want to find $\int x\ln(x)dx$, what is our u and v'?

We take $u = \ln(x)$ and $v' = x$ because $\ln(x)$ has priority over x. You can use this list to determine your u and v' of the given function. In general, you choose your u and v' such that the Right-Hand Side integral, $\int vu'dx$ of 8.45, should be simpler than the original integral, $\int uv'dx$.

Example 20

Determine $\int x\ln(x)dx$

Solution

Let

$$u = \ln(x) \qquad v' = x$$

$$u' = \frac{1}{x} \quad \text{[Differentiating]} \qquad v = \int x dx = \frac{x^2}{2} \qquad \text{[Integrating]}$$

Substituting these into the integration by parts formula:

$$\begin{aligned}\int x\ln(x)dx &= uv - \int u'v dx \\ &= \ln(x)\left(\frac{x^2}{2}\right) - \int\left(\frac{1}{x}\right)\left(\frac{x^2}{2}\right)dx \\ &= \left(\frac{x^2}{2}\right)\ln(x) - \frac{1}{2}\int x dx \quad \text{[Simplifying]} \\ &= \left(\frac{x^2}{2}\right)\ln(x) - \frac{1}{2}\left(\frac{x^2}{2}\right) + C \\ &= \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right) + C \quad \left[\text{Taking out } \frac{x^2}{2}\right]\end{aligned}$$

8.45 $\int uv'dx = uv - \int u'v dx$

D2 Engineering applications of integration by parts

Example 21 *electrical principles*

The energy, W, of an inductor is given by

$$W = \int t\cos(t)\,dt \qquad *$$

Determine W.

Solution

How do we integrate $t\cos(t)$?

Since we have a product, $t \times \cos(t)$, we try using the integration by parts formula 8.45.

What is u and v'?

By using our list we see that t is on the list whilst $\cos(t)$ is **not**. Hence

$$u = t \quad \text{and} \quad v' = \cos(t)$$

Differentiating u and integrating v' gives

$$u' = 1 \qquad v = \int \cos(t)\,dt = \sin(t)$$

Putting these into 8.45 gives

$$\begin{aligned}\int t\cos(t)dt &= uv - \int u'v\,dt \\ &= t\sin(t) - \int (1)\sin(t)dt \\ &= t\sin(t) - \int \sin(t)\,dt \\ &= t\sin(t) - \underbrace{\left[-\cos(t)\right]}_{\text{by } 8.7} + C\end{aligned}$$

$$W = \int t\cos(t)\,dt = t\sin(t) + \cos(t) + C$$

8.7 $\int \sin(t)dt = -\cos(t)$

Example 22 *mechanics*

The acceleration, $\ddot{x}$, of a particle is given by

$$\ddot{x} = te^{-t}$$

Find the velocity, $v = \dot{x}$, for the initial condition $t = 0$, $v = 0$.

Solution

$\dot{x}$ is obtained from $\ddot{x}$ by integrating $\ddot{x}$. Hence

$$v = \dot{x} = \int te^{-t}\mathrm{d}t$$

How do we integrate this function te^{-t}?

Use the integration by parts formula 8.45:

$$u = t \qquad\qquad w' = e^{-t}$$

$$u' = 1 \quad \text{[Differentiating]} \qquad w = \int e^{-t}\mathrm{d}t \underbrace{=}_{\text{by } 8.41} -e^{-t} \quad \text{[Integrating]}$$

We use w because v represents velocity in this example.

Applying 8.45:

$$v = \int te^{-t}\mathrm{d}t = uw - \int u'w\,\mathrm{d}t$$

$$= (t)(-e^{-t}) - \int (1)(-e^{-t})\mathrm{d}t$$

$$= -te^{-t} + \int e^{-t}\mathrm{d}t \quad \text{[Simplifying]}$$

$$v = -te^{-t} - e^{-t} + C$$

Substituting the given conditions $t = 0$, $v = 0$ into $v = -te^{-t} - e^{-t} + C$ yields

$$0 = 0 - 1 + C \quad \text{[Remember that } e^0 = 1\text{]}$$

$$C = 1$$

Hence putting $C = 1$ and taking out the common factor $-e^{-t}$ we have

$$v = -e^{-t}(t + 1) + 1 \text{ or } v = 1 - e^{-t}(t + 1)$$

For definite integrals the integration by parts formula is given by

8.46 $$\int_a^b uv'\mathrm{d}x = [\,uv]_a^b - \int_a^b u'v\mathrm{d}x$$

8.41 $\int e^{kt+m}\,\mathrm{d}t = e^{kt+m}/k$

Example 23 *electrical principles*

The energy stored, W, in an inductor is given by

$$W = \int_0^1 (10 \times 10^{-3}) t e^{-2t} \mathrm{d}t$$

Find the exact value of W.

Solution

By taking out the constant, (10×10^{-3}), we have

† $$W = (10 \times 10^{-3}) \int_0^1 t e^{-2t} \mathrm{d}t$$

How do we find the definite integral $\int_0^1 t e^{-2t} \mathrm{d}t$?

Since the function is a product, we try the integration by parts formula for definite integrals, 8.46. Differentiate u and integrate v':

$$u = t \qquad v' = e^{-2t}$$

$$u' = 1 \qquad v = \int e^{-2t} \mathrm{d}t = -\frac{e^{-2t}}{2} \qquad \text{[By 8.41]}$$

Thus, applying 8.46 with $a = 0$ and $b = 1$:

$$\int_0^1 t e^{-2t} \mathrm{d}t = [uv]_0^1 - \int_0^1 u' v \, \mathrm{d}t$$

$$= \left[-\frac{t e^{-2t}}{2}\right]_0^1 - \int_0^1 \left(-\frac{e^{-2t}}{2}\right) \mathrm{d}t$$

* $$= -\frac{1}{2}\Big(\underbrace{1e^{-2}}_{\substack{\text{substituting} \\ t=1}} - \underbrace{0}_{\substack{\text{substituting} \\ t=0}}\Big) + \frac{1}{2}\int_0^1 e^{-2t} \mathrm{d}t$$

We still need to integrate the last term on the Right-Hand Side of *:

$$\int_0^1 e^{-2t} \mathrm{d}t \underbrace{=}_{\text{by 8.41}} \left[\frac{e^{-2t}}{-2}\right]_0^1 = -\frac{1}{2}\left(e^{-2} - e^0\right) = -\frac{1}{2}\left(e^{-2} - 1\right)$$

8.41 $\int e^{(kt+m)} \mathrm{d}t = e^{kt+m}/k$ 8.46 $\int_a^b uv' \mathrm{d}t = [uv]_a^b - \int_a^b u'v \mathrm{d}t$

Example 23 *continued*

Substituting this into * gives:

$$\int_0^1 te^{-2t}\mathrm{d}t = -\frac{1}{2}e^{-2} + \frac{1}{2}\left[-\frac{1}{2}\left(e^{-2} - 1\right)\right]$$

$$= \frac{1}{2}\left[-e^{-2} - \frac{e^{-2}}{2} + \frac{1}{2}\right] \quad \left[\text{Taking out } \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[-\frac{3e^{-2}}{2} + \frac{1}{2}\right] \quad [\text{Simplifying}]$$

$$= \frac{1}{4}\left[1 - 3e^{-2}\right] \quad \left[\text{Taking out } \frac{1}{2}\right]$$

The exact value of W is evaluated by putting this into †:

$$W = \frac{10 \times 10^{-3}}{4}\left[1 - 3e^{-2}\right] = 2.5 \times 10^{-3}[1 - 3e^{-2}] \text{ (joule)}$$

Sometimes we need to apply the integration by parts formula, 8.46, twice as the following example shows.

Example 24 *electrical principles*

The energy stored, W, in an inductor is given by

$$W = \int_0^2 (t^2 - 2t)e^{-2t}\mathrm{d}t$$

Evaluate W.

Solution

? We need to use 8.46. **What is u and v' in this formula?**

By examining the priority list we let

$$u = t^2 - 2t \qquad v' = e^{-2t}$$

$$u' = 2t - 2 \qquad v = \int e^{-2t}\mathrm{d}t = \frac{e^{-2t}}{-2} \qquad \left[\text{Using } \int e^{kt}\,\mathrm{d}t = \frac{e^{kt}}{k}\right]$$

Substituting these into 8.46 with $a = 0$ and $b = 2$ gives

$$W = \int_0^2 (t^2 - 2t)e^{-2t}\mathrm{d}t = [uv]_0^2 - \int_0^2 u'v\,\mathrm{d}t$$

$$= \left[(t^2 - 2t)\left(\frac{e^{-2t}}{-2}\right)\right]_0^2 - \int_0^2 (2t - 2)\left(\frac{e^{-2t}}{-2}\right)\mathrm{d}t$$

$$= \underbrace{\left(\left(2^2 - (2 \times 2)\right)\frac{e^{-4}}{-2} - 0\right)}_{=0} + \frac{1}{2}\int_0^2 2(t - 1)e^{-2t}\mathrm{d}t$$

$$W = \int_0^2 (t - 1)e^{-2t}\mathrm{d}t \qquad [\text{Cancelling 2's}]$$

Example 24 *continued*

How can we find this integral?

Use 8.46 again since we have a product of $t - 1$ and e^{-2t}. Let

$$u = t - 1 \qquad v' = e^{-2t}$$

$$u' = 1 \qquad v = \int e^{-2t}\mathrm{d}t = \frac{e^{-2t}}{-2}$$

Note that these u's and v's are different from above. Putting these into the formula:

$$W = [uv]_0^2 - \int_0^2 u'\,v\,\mathrm{d}t$$

$$= \left[(t-1)\frac{e^{-2t}}{-2}\right]_0^2 - \int_0^2 (1)\frac{e^{-2t}}{-2}\mathrm{d}t$$

$$= -\frac{1}{2}\Big[(t-1)e^{-2t}\Big]_0^2 + \frac{1}{2}\int_0^2 e^{-2t}\mathrm{d}t \qquad \left[\text{Taking out } -\frac{1}{2}\right]$$

$$= -\frac{1}{2}\Big(\underbrace{(2-1)e^{-4}}_{\text{substituting } t=2} - \underbrace{(0-1)e^{0}}_{\text{substituting } t=0}\Big) + \frac{1}{2}\int_0^2 e^{-2t}\mathrm{d}t$$

$$= -\frac{1}{2}(e^{-4}+1) + \frac{1}{2}\left[\frac{e^{-2t}}{-2}\right]_0^2$$

$$= -\frac{1}{2}(e^{-4}+1) - \frac{1}{4}\Big(\underbrace{e^{-4}}_{\text{substituting } t=2} - \underbrace{e^{0}}_{\text{substituting } t=0}\Big)$$

$$= -\frac{1}{2}(e^{-4}+1) - \frac{1}{4}(e^{-4}-1) = -0.264 \qquad \text{[Evaluating]}$$

$$W = -0.264\text{ J} \quad \text{(3 d.p.)}$$

The integration by parts formula **cannot** be applied to all products of functions. For example, to find $\int \cos(2x)\sin(3x)\mathrm{d}x$ we would **not** use the integration by parts formula.

SUMMARY

Let u and v be functions of x, then

8.45 $$\int uv'\mathrm{d}x = uv - \int vu'\mathrm{d}x$$

8.45 is called the integration by parts formula and gives the integral of a product of functions.

Take u to be the function in the following order:

1 $\ln(x)$
2 x^n
3 e^{kx} where k is a constant

Exercise 8(d)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Determine the following integrals:

a $\int 2xe^x dx$ **b** $\int t\sin(t)\,dt$

2 Find **a** $\int q\cos(3q)\,dq$ **b** $\int s^2\ln(s)\,ds$

3 Obtain $\int te^{2t}\,dt$.

4 Show that

$$\int p\sqrt{1+p}\,dp = \frac{2}{15}(1+p)^{3/2}[3p-2]+C$$

5 By writing $\ln(x) = 1 \times \ln(x)$, show that

$$\int \ln(x)\,dx = x[\ln(x)-1]+C$$

6 *[electrical principles]* The current, i, through an inductor with inductance L is given by

$$i = \frac{1}{L}\int_0^t v\,dt$$

where v is the voltage across the inductor. For a circuit with $L = 10 \times 10^{-3}$ H and $v = 5te^{-t}$, find i.

7 *[electrical principles]* The energy, w, of an inductor with inductance L is given by

$$w = \frac{1}{2L}\left(\int_0^t v\,dt\right)^2$$

where v is the voltage across the inductor. For $L = 10$ mH and $v = t\cos(t)$, find w.

8 *[electrical principles]* The current, i, through an inductor with inductance L is given by

$$i = \frac{1}{L}\int_0^1 v\,dt$$

For $L = (1 \times 10^{-3})$ H and $v = t^2 e^{-t}$, find i.

SECTION E **Algebraic fractions**

By the end of this section you will be able to:

- ▶ understand what is meant by the term partial fraction
- ▶ find partial fractions of various expressions
- ▶ understand the procedure for finding partial fractions
- ▶ find partial fractions of improper fractions

Partial fractions are an algebraic topic which can be used to integrate algebraic fractions. **For example, how do we integrate the following:**

$$\int \frac{x}{(x+2)(x+1)}dx?$$

We can't use any of the techniques discussed earlier. We are compelled to express $\frac{x}{(x+2)(x+1)}$ as partial fractions and then integrate the result. This section could have been placed in the earlier Chapters 1 and 2 on algebra, however partial fractions is a difficult topic and requires greater algebraic skills.

E1 Partial fractions

Consider the following example:

$$\frac{7}{12} = \frac{1}{4} + \frac{1}{3}$$

We say that $\frac{1}{4}$ and $\frac{1}{3}$ are partial fractions of $\frac{7}{12}$. The two or more fractions which create a single fraction are called **partial fractions**.

Consider the following algebraic expression:

$$\frac{x}{(x+2)(x+1)} = \frac{2}{x+2} - \frac{1}{x+1}$$

? **What are the partial fractions of $\frac{x}{(x+2)(x+1)}$?**

Clearly they are $\frac{2}{x+2}$ and $\frac{-1}{x+1}$.

In this section we study the processes of splitting a fraction such as $\frac{x}{(x+2)(x+1)}$ into its partial fractions.

? **How do we split $\frac{x}{(x+2)(x+1)}$ into its partial fractions?**

First we write this as

† $$\frac{x}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

where A and B are constants which we need to find.

? **How do we know that $\frac{x}{(x+2)(x+1)}$ breaks into $\frac{A}{x+2} + \frac{B}{x+1}$?**

Well, if we add the fractions

$$\frac{A}{x+2} + \frac{B}{x+1}$$

then we obtain a common denominator of $(x + 2)(x + 1)$. Thus

$$\frac{A}{x+2} + \frac{B}{x+1} = \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$$

So all we are left to achieve is to find the constants A and B so that the numerator gives an x.

†† $$\frac{x}{(x+2)(x+1)} = \frac{A(x+1) + B(x+2)}{(x+2)\,(x+1)}$$

Since on both sides of the equal sign we have a common denominator, the numerator must be the same, that is

* $$x = A(x+1) + B(x+2)$$

This can also be attained by multiplying both sides of †† by $(x + 2)(x + 1)$.

But how can we find A and B?

We can choose values of x and substitute these values into * . The evaluations of A and B are simpler if we select our x values such that some of the terms on the Right-Hand Side of * vanish. For example, choosing $x = -1$ gives

$$-1 = A(-1 + 1) + B(-1 + 2)$$
$$-1 = 0 + B$$

Hence $B = -1$.

How can we find A?

By choosing $x = -2$ and substituting into * :

$$-2 = A(-2 + 1) + B(-2 + 2)$$
$$-2 = A(-1) + 0$$

Hence $A = 2$.

Why did we choose $x = -2$?

Because this removes the B term of * .

Putting $A = 2$ and $B = -1$ into † gives the identity

$$\frac{x}{(x+2)(x+1)} = \frac{2}{x+2} - \frac{1}{x+1}$$

We have broken the single fraction, $\frac{x}{(x+2)(x+1)}$, into the difference of two fractions, $\frac{2}{x+2} - \frac{1}{x+1}$. This is what we started with at the beginning of this discussion.

For the appropriate values of the constants, the partial fractions become an identity; that is, both sides are equal for all values of x.

Let's investigate another example.

Example 25

Resolve

$$\frac{2x}{x^2 - x - 2}$$

into partial fractions.

Solution

We first try to place the denominator in the form of two bracketed terms, that is to factorize the denominator $x^2 - x - 2$. This factorizes into

$$x^2 - x - 2 = (x + 1)(x - 2)$$

Therefore

$$\frac{2x}{x^2 - x - 2} = \frac{2x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

Multiply both sides by $(x + 1)(x - 2)$:

$$2x = \frac{A(x+1)(x-2)}{(x+1)} + \frac{B(x+1)(x-2)}{(x-2)}$$
$$= A(x - 2) + B(x + 1) \quad \text{[Cancelling]}$$

We have

* $$2x = A(x - 2) + B(x + 1)$$

Example 25 *continued*

Which values of x should we select to find A and B?

Putting $x = 2$ into * removes the A term:

$$2 \times 2 = A(2 - 2) + B(2 + 1)$$
$$4 = 3B$$
$$B = \frac{4}{3}$$

To find A, put $x = -1$ into * which removes the B term:

$$2 \times (-1) = A(-1 - 2) + B(0)$$
$$-2 = -3A$$
$$A = \frac{2}{3}$$

Putting $A = \frac{2}{3}$ and $B = \frac{4}{3}$ into

$$\frac{2x}{x^2 - x - 2} = \frac{A}{x + 1} + \frac{B}{x - 2}$$

gives

** $$\frac{2x}{x^2 - x - 2} = \frac{\frac{2}{3}}{x + 1} + \frac{\frac{4}{3}}{x - 2}$$

Consider the first term on the Right-Hand Side of **:

$$\frac{\frac{2}{3}}{x + 1} = \frac{2}{3} \div (x + 1) = \frac{2}{3} \div \frac{(x + 1)}{1} = \frac{2}{3} \times \frac{1}{(x + 1)}$$
$$= \frac{2 \times 1}{3 \times (x + 1)} = \frac{2}{3(x + 1)}$$

Similarly the second term on the Right-Hand Side of ** becomes

$$\frac{\frac{4}{3}}{x - 2} = \frac{4}{3(x - 2)}$$

Combining these two results we have the identity

$$\frac{2x}{x^2 - x - 2} = \frac{2}{3(x + 1)} + \frac{4}{3(x - 2)}$$
$$= \frac{2}{3}\left[\frac{1}{x + 1} + \frac{2}{x - 2}\right] \qquad \left[\text{Taking out } \frac{2}{3}\right]$$

Consider a single algebraic fraction

8.47 $$\frac{f(x)}{g(x)}$$

where $g(x)$ factorizes and is of a higher degree polynomial than $f(x)$. **What do we mean by a higher degree polynomial?**

For example, if $g(x)$ is a cubic (containing x^3) then $f(x)$ is at most a quadratic (containing x^2). See Table 3.

TABLE 3

Polynomial	*Degree*	*Name*
$x - 5$	1	linear
$x^2 + x + 1$	2	quadratic
$x^3 - x^2$	3	cubic
$x^4 + x - 3$	4	quartic

We need to write 8.47, $\frac{f(x)}{g(x)}$, as a sum of partial fractions. This can be obtained by factorizing the denominator, $g(x)$, and then using one of the following rules.

Each linear factor of the denominator has partial fractions of the form $\frac{A}{ax + b}$. Thus

8.48 $$\frac{f(x)}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}$$

Each repeated linear factor of the denominator has partial fractions of the form $\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$. Thus

8.49 $$\frac{f(x)}{(ax + b)^2} = \frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

Each quadratic factor of the denominator has the partial fractions of the form $\frac{Ax + B}{ax^2 + bx + c}$. Hence

8.50 $$\frac{f(x)}{ax^2 + bx + c} = \frac{Ax + B}{ax^2 + bx + c}$$

A combination of linear and quadratic factors gives

8.51 $$\frac{f(x)}{(ax^2 + bx + c)(dx + e)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{C}{dx + e}$$

For the appropriate values of A, B and C, 8.48 to 8.51 are identities.

These identities look horrendous but once we do a few examples then you will become familiar with these 'horrendous' identities. However the procedure in finding the partial fractions of

8.47 $$\frac{f(x)}{g(x)}$$

is

1 Factorize the denominator, $g(x)$, as far as possible.
2 Write 8.47 as one of the general partial fractions by choosing the appropriate identity from 8.48 to 8.51.
3 Find the values of the unknown constants A, B, C, etc.

Let's do an example.

Example 26

Express

$$\frac{3t^2 - t + 1}{(t^2 - t + 3)(t - 1)}$$

as partial fractions.

Solution

The denominator does not factorize further into simpler factors. Each factor of the denominator, $t^2 - t + 3$ and $t - 1$, produces a partial fraction. **Which identity, 8.48 to 8.51, is appropriate for**

$$\frac{3t^2 - t + 1}{(t^2 - t + 3)(t - 1)}?$$

Since we have quadratic and linear factors we use 8.51:

† $$\frac{3t^2 - t + 1}{(t^2 - t + 3)(t - 1)} = \frac{At + B}{t^2 - t + 3} + \frac{C}{t - 1}$$

What do we need to find?

The constants A, B and C. **How can we find A, B and C?**

Multiplying both sides of † by $(t^2 - t + 3)(t - 1)$ we have

†† $$3t^2 - t + 1 = (At + B)(t - 1) + C(t^2 - t + 3)$$

To remove the first term on the Right-Hand Side of †† we substitute $t = 1$ into ††:

$$3 - 1 + 1 = 0 + C(1^2 - 1 + 3)$$
$$3 = 3C \text{ gives } C = 1$$

Putting $C = 1$ into †† yields

* $$3t^2 - t + 1 = (At + B)(t - 1) + C(t^2 - t + 3)$$

How can we find A and B?

We can now substitute other values of t such as $t = 0$, but it is generally easier to **equate coefficients.** Conventionally we start to equate coefficients of the highest power first. So we first equate the number of t^2 on the left of * to the number of t^2 on the right of *.

Example 26 *continued*

We need to expand the Right-Hand Side of * because $At \times t$ gives At^2 and so the number of t^2 on the right is $A + 1$.

? **How many t^2 are on the Left-Hand Side of * ?**

3. Thus equating coefficients of t^2 we have

$$3 = A + 1 \text{ therefore } A = 2$$

Equating coefficients of t in * by expanding the Right-Hand Side of * gives

$$At \times (-1) = -At \text{ and } B \times t = Bt$$

$$-1 = -A + B - 1 \qquad [\text{Equating } t\text{'s of } *]$$

Substituting $A = 2$ yields

$$-1 = -2 + B - 1$$
$$-1 = -3 + B \text{ gives } B = 2$$

So we have $A = 2,\ B = 2$ and $C = 1$. Substituting these into † displays the partial fractions

$$\frac{3t^2 - t + 1}{(t^2 - t + 3)(t - 1)} = \frac{2t + 2}{t^2 - t + 3} + \frac{1}{t - 1}$$

E2 Improper fractions

If in the algebraic fraction

$$\frac{f(x)}{g(x)}$$

$f(x)$ is a polynomial of higher, or same, degree compared with $g(x)$ then $\frac{f(x)}{g(x)}$ is called an **improper fraction.** The degree of a polynomial, $f(x)$, is the highest power of x contained in the polynomial.

To find the partial fractions of an improper fraction, $\frac{f(x)}{g(x)}$, we first divide $f(x)$ by $g(x)$ by applying long division. If this division results in a polynomial $q(x)$ with remainder $R(x)$ then we can write

8.52 $$\frac{f(x)}{g(x)} = q(x) + \frac{R(x)}{g(x)}$$

where the remainder $R(x)$ is a polynomial of lower degree than $g(x)$. We can now treat $\frac{R(x)}{g(x)}$ in the usual way for partial fractions.

Let's first do an example on arithmetic long division.

Applying long division to $200 \div 15$ we have

$$\begin{array}{r} 13 \\ 15\,\overline{)200} \\ \underline{195} \\ 5 \end{array} \quad \text{(Subtracting 195 from 200)}$$

This says that 15 goes 13 whole times into 200 with remainder 5:

$$\frac{200}{15} = 13 + \frac{5}{15}$$

Example 27

Express $\dfrac{x^3}{x^2 - 4}$ in partial fractions.

Solution

Since x^3 is of degree 3 and $x^2 - 4$ is of degree 2 we first apply long division.

Long division of algebraic expressions is the same procedure as long division of numbers. To obtain an x^3 from $x^2 - 4$ we multiply by x.

$x(x^2 - 4) = x^3 - 4x$, thus

$$\begin{array}{r|l} & x \\ x^2 - 4 & x^3 \\ & x^3 - 4x \\ \hline & 0 + 4x \end{array} \quad \text{(Subtracting } x^3 - 4x \text{ from } x^3\text{)}$$

Using 8.52 with $q(x) = x$ and $R(x) = 4x$ we have

† $$\frac{x^3}{x^2 - 4} = x + \frac{4x}{x^2 - 4}$$

8.52 puts the expression $\dfrac{x^3}{x^2 - 4}$ into partial fractions but we can cut it into even more partial fractions because the denominator, $x^2 - 4$, factorizes. Thus

$$x^2 - 4 = x^2 - 2^2 = (x - 2)(x + 2)$$

? **Which identity, 8.48 to 8.51, can we use to place $\dfrac{4x}{(x - 2)(x + 2)}$ into partial fractions?**

Use

8.48 $$\frac{f(x)}{(ax + b)(cx + d)} = \frac{A}{ax + b} + \frac{B}{cx + d}$$

†† $$\frac{4x}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2}$$

Multiplying both sides by $(x - 2)(x + 2)$ gives

* $$4x = A(x + 2) + B(x - 2)$$

? **What are we trying to find?**

The values of the constants A and B.

? **What should we substitute for x into * to obtain A and B?**

Putting $x = 2$ into * gives

$$(4 \times 2) = A(2 + 2) + 0$$
$$8 = 4A \text{ which gives } A = 2$$

Example 27 *continued*

Putting $x = -2$ into * gives

$$[4 \times (-2)] = 0 + B(-2 - 2)$$
$$-8 = -4B \text{ which gives } B = 2$$

Substituting $A = 2,\ B = 2$ into †† yields

$$\frac{4x}{(x-2)(x+2)} = \frac{2}{x-2} + \frac{2}{x+2} \underbrace{=}_{\text{taking out the common factor 2}} 2\left[\frac{1}{x-2} + \frac{1}{x+2}\right]$$

Putting $\frac{4x}{x^2-4} = 2\left[\frac{1}{x-2} + \frac{1}{x+2}\right]$ into † gives the identity

$$\frac{x^3}{x^2-4} = x + 2\left[\frac{1}{x-2} + \frac{1}{x+2}\right]$$

SUMMARY

The procedure for finding partial fractions of

8.47 $$\frac{f(x)}{g(x)}$$

where $g(x)$ factorizes and is of a higher degree than $f(x)$ is

1. factorize the denominator
2. write 8.47 as one of the general partial fractions determined by 8.48 to 8.51
3. find the values of the unknown constants A, B, C, etc.

If $f(x)$ is of the same or higher degree than $g(x)$ in 8.47 then this is called an improper fraction and we need to apply long division.

Exercise **8(e)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Express the following in partial fractions:

a $\frac{3x+4}{(x+1)(x+2)}$ **b** $\frac{2t}{t^2-1}$

c $\frac{2s+7}{s^2+s-2}$ **d** $\frac{-12u-13}{(2u+1)(u-3)}$

2 Resolve the following into partial fractions:

a $\frac{x^2}{x+1}$ **b** $\frac{x^5-2x^2}{x^2-1}$

3 Resolve the following into partial fractions:

a $\frac{4t^2+t-3}{(t^2+t-1)(t-1)}$

b $\frac{z+1}{(z-1)^2}$

c $\frac{2x^3+3x^2+5x+2}{(x^2+x+1)^2}$

SECTION F **Integration of algebraic fractions**

By the end of this section you will be able to:

- integrate $\dfrac{f(x)}{g(x)}$ by using partial fractions

F1 **Integration by partial fractions**

In this section we integrate $\dfrac{f(x)}{g(x)}$ by expressing this in its partial fractions and then integrating. Generally you will discover that once we have found the partial fractions then to integrate we employ formula 8.42:

8.42 $$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Example 28

Determine $\displaystyle\int \frac{s}{(s+2)(s+1)}\,ds$

Solution

First we express $\dfrac{s}{(s+2)(s+1)}$ in partial fractions. This was obtained in **Section E1** with x in place of s:

$$\frac{s}{(s+2)(s+1)} = \frac{2}{s+2} - \frac{1}{s+1}$$

? **How do we integrate $\dfrac{s}{(s+2)(s+1)}$ with respect to s?**

$$\begin{aligned}
\int \frac{s}{(s+2)(s+1)}\,ds &= \int \left(\frac{2}{s+2} - \frac{1}{s+1}\right)ds \\
&= \int \frac{2ds}{s+2} - \int \frac{ds}{s+1} && \text{[Splitting integrand]} \\
&= 2\int \frac{ds}{s+2} - \int \frac{ds}{s+1} && \text{[Taking out 2]} \\
&\underset{\text{by } 8.42}{=} 2\ln|s+2| - \ln|s+1| + C && \text{[Integrating]} \\
&= \underbrace{\ln(|s+2|^2)}_{\text{by } 5.14} - \ln|s+1| + C \\
&= \ln\left|\frac{(s+2)^2}{s+1}\right| + C && \text{[By 5.13]}
\end{aligned}$$

5.13 $\ln(A) - \ln(B) = \ln(A/B)$ 5.14 $n\ln(A) = \ln(A^n)$

Note that to simplify the result of integration we apply the laws of logarithms. This is generally the case for integration by partial fractions and so make sure that the laws of logarithms, 5.12 to 5.14, are second nature to you.

Example 29

Find $\int \frac{2x}{x^2 - x - 2}\,dx$

Solution

We have already placed the integrand, $\frac{2x}{x^2 - x - 2}$, in partial fractions in **Example 25**, thus

$$\frac{2x}{x^2 - x - 2} = \frac{2}{3}\left[\frac{1}{x+1} + \frac{2}{x-2}\right]$$

Carrying out the integration by splitting the integrand and taking out the constant, 2/3, we have

$$\begin{aligned}
\int \frac{2x}{x^2 - x - 2}\,dx &= \frac{2}{3}\int\left[\frac{1}{x+1} + \frac{2}{x-2}\right]dx \\
&= \frac{2}{3}\left[\int \frac{1}{x+1}\,dx + \int \frac{2}{x-2}\,dx\right] \\
&= \frac{2}{3}\Big[\ln|x+1| + 2\ln|x-2|\Big] + C \\
&= \frac{2}{3}\Big[\ln|x+1| + \underbrace{\ln|x-2|^2}_{\text{by } 5.14}\Big] + C
\end{aligned}$$

The next example is long but not difficult. First we need to find the partial fractions and then integrate.

Example 30 *mechanics*

The velocity, v, of an object falling in air at time t is given by

$$t = \int_0^v \frac{dv}{9 - 0.25v^2}$$

By using partial fractions, show that

$$t = \frac{1}{3}\ln\left|\frac{3 + 0.5v}{3 - 0.5v}\right|$$

5.14 $n\ln(A) = \ln(A^n)$

Example 30 *continued*

Is there a more straightforward way to obtain this result?

Solution

First we need to factorize the denominator, $9 - 0.25v^2$:

$$9 - 0.25v^2 = 3^2 - (0.5v)^2 = (3 - 0.5v)(3 + 0.5v)$$

By placing into partial fractions we have

† $$\frac{1}{9 - 0.25v^2} = \frac{1}{(3 - 0.5v)(3 + 0.5v)} \underset{\text{by 8.48}}{=} \frac{A}{3 - 0.5v} + \frac{B}{3 + 0.5v}$$

Multiplying both sides by $(3 - 0.5v)(3 + 0.5v)$ gives

* $$1 = A(3 + 0.5v) + B(3 - 0.5v)$$

How can we find A and B?

Put $v = -6$ into * because this will remove the A term and therefore we can find B:

$$1 = A[3 + 0.5(-6)] + B[3 - 0.5(-6)]$$
$$1 = 0 + 6B$$

Thus $B = \frac{1}{6}$. Putting $v = 6$ into * gives

$$1 = A[3 + (0.5 \times 6)] + 0$$
$$1 = 6A$$

So $A = \frac{1}{6}$. Substituting $A = \frac{1}{6}$ and $B = \frac{1}{6}$ into † gives the partial fractions

$$\frac{1}{9 - 0.25v^2} = \frac{\frac{1}{6}}{3 - 0.5v} + \frac{\frac{1}{6}}{3 + 0.5v}$$
$$= \frac{1}{6}\left[\frac{1}{3 - 0.5v} + \frac{1}{3 + 0.5v}\right] \quad \left[\text{Taking out } \frac{1}{6}\right]$$

Considering the given integral we have

$$t = \int_0^v \frac{dv}{9 - 0.25v^2} = \frac{1}{6}\int_0^v \left(\frac{1}{3 - 0.5v} + \frac{1}{3 + 0.5v}\right)dv$$

†† $$t = \frac{1}{6}\left[\int_0^v \frac{dv}{3 - 0.5v} + \int_0^v \frac{dv}{3 + 0.5v}\right]$$

8.48 $$\frac{f(v)}{(av + b)(cv + d)} = \frac{A}{av + b} + \frac{B}{cv + d}$$

Example 30 *continued*

How do we find the first integral on the Right-Hand Side, $\int_0^v \frac{dv}{3 - 0.5v}$?

We can spot that differentiating the denominator gives −0.5. If we want to use

8.42 $$\int \frac{f'(v)}{f(v)}\,dv = \ln|f(v)| + C$$

then we need to have the derivative of the denominator on the numerator.

We can rewrite the integrand as follows:

$$\frac{1}{3 - 0.5v} = \frac{1}{-0.5}\left(\frac{-0.5}{3 - 0.5v}\right) = -2\left(\frac{-0.5}{3 - 0.5v}\right)$$

Evaluating the integral:

$$\begin{aligned}\int_0^v \frac{dv}{3 - 0.5v} &= -2\int_0^v \frac{-0.5}{3 - 0.5v}\,dv \\ &= -2\Big[\underbrace{\ln|3 - 0.5v|}_{\text{by } 8.42}\Big]_0^v \\ &= -2\big[\ln|3 - 0.5v| - \ln(3)\big] \quad \text{[Substituting]}\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^v \frac{dv}{3 + 0.5v} &= \frac{1}{0.5}\int_0^v \frac{0.5dv}{3 + 0.5v} \\ &= \frac{1}{0.5}\Big[\ln|3 + 0.5v|\Big]_0^v \quad \text{[Integrating]} \\ &= 2\big[\ln|3 + 0.5v| - \ln(3)\big] \quad \text{[Substituting]}\end{aligned}$$

Substituting these results into †† produces

$$\begin{aligned}t &= \frac{1}{6}\Big[-2\big[\ln|3 - 0.5v| - \ln(3)\big] + 2\big[\ln|3 + 0.5v| - \ln(3)\big]\Big] \\ &= \frac{2}{6}\Big[\ln(3) - \ln|3 - 0.5v| + \ln|3 + 0.5v| - \ln(3)\Big] \quad \text{[Taking out 2]} \\ &= \frac{1}{3}\Big[\ln|3 + 0.5v| - \ln|3 - 0.5v|\Big] \\ &= \frac{1}{3}\underbrace{\ln\left|\frac{3 + 0.5v}{3 - 0.5v}\right|}_{\text{by } 5.13} \quad \text{(Simplifying)}\end{aligned}$$

However, there is an effortless way of obtaining the above result. We can use

8.30 $$\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a + u}{a - u}\right|$$

(this is given in Table 1). Thus

$$\int_0^v \frac{dv}{9 - 0.25v^2} = \int_0^v \frac{dv}{3^2 - (0.5v)^2}$$

5.13 $\ln(A) - \ln(B) = \ln(A/B)$ 8.42 $\int \frac{f'(v)}{f(v)}\,dv = \ln|f(v)|$

Example 30 *continued*

Let $u = 0.5v$, then

$$\frac{du}{dv} = 0.5 \text{ gives } dv = \frac{du}{0.5} = 2du$$

We can first evaluate the integral without the limits:

$$\int \frac{dv}{3^2 - (0.5v)^2} = \int \frac{2du}{3^2 - u^2} = 2\int \frac{du}{3^2 - u^2}$$

$$= 2\left[\frac{1}{2 \times 3}\ln\left|\frac{3+u}{3-u}\right|\right] \quad \text{[By 8.30]}$$

$$= \frac{1}{3}\ln\left|\frac{3+u}{3-u}\right| \quad \text{[Cancelling 2's]}$$

$$= \frac{1}{3}\ln\left|\frac{3+0.5v}{3-0.5v}\right| \quad \text{[Remember that } u = 0.5v\text{]}$$

Using the limits of integration:

$$\int_0^v \frac{dv}{9 - 0.25v^2} = \frac{1}{3}\left[\ln\left|\frac{3+0.5v}{3-0.5v}\right|\right]_0^v$$

$$= \frac{1}{3}\left[\ln\left|\frac{3+0.5v}{3-0.5v}\right| - \ln\left|\frac{3}{3}\right|\right] \quad \text{[Substituting]}$$

$$= \frac{1}{3}\ln\left|\frac{3+0.5v}{3-0.5v}\right| \quad \left[\text{Because } \ln\left|\frac{3}{3}\right| = \ln|1| = 0\right]$$

SUMMARY

Integration by partial fractions involves first putting the function into partial fractions and then integrating. The most important integral formula for these fractions is 8.42. Generally we need to use the laws of logarithms to simplify.

Exercise 8(f)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Determine

a $\int \frac{3c+4}{(c+1)(c+2)}\,dc$ **b** $\int \frac{2\lambda}{\lambda^2 - 1}\,d\lambda$

c $\int \frac{2a+7}{a^2+a-2}\,da$

d $\int \frac{-12y-13}{(2y+1)(y-3)}\,dy$

2 Determine

a $\int \frac{4p^2+p-3}{(p^2+p-1)(p-1)}\,dp$

b $\int \frac{z+1}{(z-1)^2}\,dz$

3 Evaluate $\int_1^2 \frac{5z^2}{(z^2+1)(2z-1)}\,dz$

8.30 $\int \frac{du}{a^2 - u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|$

Exercise 8(f) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

4 [*mechanics*] The velocity, v, of an object moving in a medium at time t is given by

$$t = \int_{10}^{100} \frac{dv}{v(2v + 1)}$$

Evaluate t.

5 [*mechanics*] The velocity, v, of an object falling in air is given by

$$v = \int \frac{dv}{1 - (kv)^2}$$

where k is a non-zero constant. By using partial fractions, show that

$$v = \frac{1}{2k}\ln\left|\frac{1 + kv}{1 - kv}\right| + C$$

6 Show that $\int_0^1 \frac{5 + 2x - x^2}{(x^2 + 1)(x + 1)}\,dx = \pi$.

7 **i** Express $\frac{x^3 + 1}{x^2 + 3x + 2}$ in partial fractions.

ii Determine $\int \frac{x^3 + 1}{x^2 + 3x + 2}\,dx$

SECTION G Integration by substitution revisited

By the end of this section you will be able to:

- ▶ evaluate integrals by substitution
- ▶ integrate $k\left[f(t)\right]^n f'(t)$

G1 Integration by substitution

In **Section B** of this chapter we found indefinite integrals (integrals without limits) by using a substitution. **Why did we use a substitution?**

The integral of interest was not among the list of standard integrals of Table 1. Additionally, the evaluation of the integral became elementary once we used the substitution.

Since then we have developed other methods of integration such as integration by parts and by partial fractions. In this section we examine integrals with limits which demand a substitution. Let's do an example.

Example 31

Evaluate $\int_1^4 \frac{t}{\sqrt{3t^2 + 1}}\,dt$.

Solution

Can you identify a substitution that we can use to evaluate this integral?

One choice might be to let $u = 3t^2 + 1$ because differentiating this we have

$$\frac{du}{dt} = 6t$$

Example 31 *continued*

which gives

$$\frac{du}{6t} = dt$$

Therefore we might be able to cancel the t's once substitution has taken place.

? **We also need to substitute new values for the limits $t = 1$ and $t = 4$. Why?**

Because we are integrating with respect to a new variable, $u = 3t^2 + 1$.

When $t = 1, \quad u = (3 \times 1^2) + 1 = 4$

When $t = 4, \quad u = (3 \times 4^2) + 1 = 49$

Substituting the new limits and $u = 3t^2 + 1, \; dt = \dfrac{du}{6t}$, we have

$$\int_1^4 \left(\frac{t}{\sqrt{3t^2+1}}\right) dt = \int_{u=4}^{u=49} \left(\frac{t}{\sqrt{u}}\right)\frac{du}{6t}$$

$$= \frac{1}{6}\int_4^{49} \frac{du}{u^{1/2}} \quad \text{[Cancelling } t\text{'s]}$$

$$= \frac{1}{6}\int_4^{49} u^{-1/2} du$$

$$= \frac{1}{6}\left[\frac{u^{1/2}}{1/2}\right]_4^{49} \underbrace{=}_{\substack{\text{substituting}\\ \text{the limits}}} \frac{1}{3}\left[49^{1/2} - 4^{1/2}\right] = \frac{5}{3}$$

Notice that the t's in the above example cancel, and for this reason the integration becomes straightforward. In general, to integrate a function of the type

8.53 $\quad k\,[f(t)]^n f'(t) \quad$ (where $n \neq -1$ and k is a constant)

with respect to t, we use the substitution $u = f(t)$.

8.53 might look terrifying but it is only asserting that if you have a function, $f(t)$, to the power n and its derivative multiplied by a constant in the form of 8.53, then use the substitution $u = f(t)$. For **Example 31**:

$$f(t) = 3t^2 + 1, \; f'(t) = 6t, \; k = \frac{1}{6} \text{ and } n = -\frac{1}{2}$$

$$\frac{1}{6}(3t^2+1)^{-1/2}\,6t = \frac{t}{\sqrt{3t^2+1}}$$

Let's explain this further by trying another example.

Example 32

Determine

$$\int \frac{t}{(t^2+1)^5}\,\mathrm{d}t$$

Solution

If we let $u = t^2 + 1$, differentiating gives

$$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$$

(In relation to 8.53, we have $f(t) = t^2 + 1$ and $f'(t) = 2t$.)

Rearranging $\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$ gives

$$\mathrm{d}t = \frac{\mathrm{d}u}{\mathrm{d}u/\mathrm{d}t} = \frac{\mathrm{d}u}{2t}$$

Remember that we need to replace the dt in the original integral because we are integrating with respect to u. Thus, substituting $u = t^2 + 1$ and $\mathrm{d}t = \frac{\mathrm{d}u}{2t}$ gives

$$\begin{aligned}
\int \left(\frac{t}{(t^2+1)^5}\right)\mathrm{d}t &= \int \left(\frac{t}{u^5}\right)\frac{\mathrm{d}u}{2t} \\
&= \frac{1}{2}\int \frac{\mathrm{d}u}{u^5} \quad \text{[Cancelling } t\text{'s]} \\
&= \frac{1}{2}\int u^{-5}\mathrm{d}u \\
&= \frac{1}{2}\left(\frac{u^{-4}}{-4}\right) + C = -\frac{u^{-4}}{8} + C = C - \frac{1}{8u^4}
\end{aligned}$$

We have

$$\int \frac{t}{(t^2+1)^5}\,\mathrm{d}t = C - \frac{1}{8(t^2+1)^4} \qquad \text{[Substituting } u = t^2 + 1\text{]}$$

G2 Some trigonometric substitutions

The method of substitution is important when integrating trigonometric functions.

Example 33

Evaluate $\int_0^{\frac{\pi}{2}} \left(\cos(\alpha)\sqrt{\sin(\alpha)}\right)\mathrm{d}\alpha$

Example 33 *continued*

Solution

When we differentiate $\sin(\alpha)$ we obtain $\cos(\alpha)$.

Thus the function $\cos(\alpha)\sqrt{\sin(\alpha)}$ seems to be of the form of 8.53. So use the substitution $u = \sin(\alpha)$. Differentiating gives

$$\frac{du}{d\alpha} = \cos(\alpha)$$

$$d\alpha = \frac{du}{\cos(\alpha)}$$

We also need to change the limits of integration:

When $\alpha = 0$, $\quad u = \sin(0) = 0$

When $\alpha = \frac{\pi}{2}$, $\quad u = \sin\left(\frac{\pi}{2}\right) = 1$

Replacing the limits and $u = \sin(\alpha)$, $d\alpha = \frac{du}{\cos(\alpha)}$, into the original integral we have

$$\int_{\alpha=0}^{\alpha=\frac{\pi}{2}} \left(\cos(\alpha)\sqrt{\sin(\alpha)}\right) d\alpha = \int_{u=0}^{u=1} \left(\cos(\alpha)\sqrt{u}\right) \frac{du}{\cos(\alpha)}$$

$$= \int_0^1 \sqrt{u}\, du \quad [\text{Cancelling } \cos(\alpha)]$$

$$= \int_0^1 u^{1/2} du$$

$$= \left[\frac{u^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}\left[u^{3/2}\right]_0^1 = \frac{2}{3}\left(1^{3/2} - 0^{3/2}\right) = \frac{2}{3}$$

$$\int_0^{\frac{\pi}{2}} \left(\cos(\alpha)\sqrt{\sin(\alpha)}\right) d\alpha = \frac{2}{3}$$

SUMMARY

To integrate a function of the type

8.53 $\quad k[f(t)]^n f'(t) \quad$ (where $n \neq -1$ and k is a constant)

with respect to t, use the substitution $u = f(t)$.

For a definite integral we also need to replace the limits of integration:

$$\int_{x=a}^{x=b} f(x)dx = \int_{u=c}^{u=d} g(u)du$$

Exercise 8(g)

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 Determine the following integrals:

a $\int b(b^2 - 3)^7 \mathrm{d}b$

b $\int (5s - 1)^9 \mathrm{d}s$

c $\int (3a^2 - 4a)(a^3 - 2a^2 + 6)^4 \, \mathrm{d}a$

d $\int 21q^2\sqrt{7q^3 - 5} \, \mathrm{d}q$

e $\int \frac{p^2 - 1}{\sqrt{p^3 - 3p}} \, \mathrm{d}p$

f $\int \frac{\alpha - 1}{(\alpha^2 - 2\alpha + 10)^2} \, \mathrm{d}\alpha$

2 [*reliability engineering*] The mean time to failure, MTTF, in years, for a set of components is given by

$$\text{MTTF} = \int_0^5 (1 - 0.2t)^{1.5} \mathrm{d}t$$

Evaluate MTTF.

3 Determine $\int_0^1 xe^{-x^2} \mathrm{d}x$. [Consider $u = x^2$]

4 Evaluate the following integrals:

a $\int_0^{\frac{\pi}{2}} \sin(\theta)\sqrt{\cos(\theta)} \, \mathrm{d}\theta$

b $\int_0^{\pi} \sin(\theta) \cos^5(\theta)\mathrm{d}\theta$

What do you notice about your result for **a**?

5 Determine the following indefinite integrals:

a $\int \sec^7(\beta)\tan(\beta)\mathrm{d}\beta$

b $\int \tan^5(A)\sec^2(A)\mathrm{d}A$

c $\int \cot^3(A)\mathrm{cosec}(A)\mathrm{d}A$

[Consider $u = \mathrm{cosec}(A)$]

SECTION H **Trigonometric techniques for integration**

By the end of this section you will be able to:

- use trigonometric substitutions to integrate trigonometric functions
- show some integral results of Table 1

This is a challenging section which requires knowledge of trigonometric identities, integration and substitution. Go through each example very carefully.

H1 **Trigonometric substitutions**

? How do we integrate $\cos^2(t)$ with respect to t?

? Can we use integration by parts because $\cos^2(t) = \cos(t) \times \cos(t)$?

No, because if we use integration by parts then we will end up in a vicious circle.

We need to substitute something for $\cos^2(t)$. In Chapter 4 we had

4.67 $$\cos^2(t) = \frac{1}{2}\left[1 + \cos(2t)\right]$$

Of course this identity might **not** have occurred to you since it is among many trigonometric identities in Chapter 4. However you can derive this identity from the fundamental identity:

$$\cos(2t) = \cos^2(t) - \sin^2(t) = 2\cos^2(t) - 1$$

Example 34

Determine $\int \cos^2(t)\,dt$.

Solution

Using **4.67** we have

$$\int \cos^2(t)\,dt = \int \frac{1}{2}\left[1 + \cos(2t)\right]dt$$

$$= \frac{1}{2}\int\left[1 + \cos(2t)\right]dt \quad \left[\text{Taking out } \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[t + \frac{\sin(2t)}{2}\right] + C \quad \left[\text{By } \int \cos(kt)\,dt = \frac{\sin(kt)}{k}\right]$$

For these types of trigonometric functions, we need to use the appropriate identity from Chapter 4. Let's investigate an engineering example.

Example 35 *electrical principles*

The voltage, v, and current, i, across a pure capacitance is given by

$$v = V\sin(\omega t) \text{ and } i = I\cos(\omega t)$$

where V is the peak voltage, I is the peak current, ω is the angular frequency and t is time. The average power, p, is given by

$$p = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(vi)\,dt$$

Show that $p = 0$.

Solution

Substituting $v = V\sin(\omega t)$ and $i = I\cos(\omega t)$ into p produces:

$$p = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\left[V\sin(\omega t)\, I\cos(\omega t)\right]dt$$

***** $$= \frac{\omega VI}{2\pi}\int_0^{2\pi/\omega}\left[\sin(\omega t)\cos(\omega t)\right]dt$$

Example 35 *continued*

The problem is how do we integrate $\sin(\omega t)\cos(\omega t)$?

Using

4.53
$$2\sin(A)\cos(A) = \sin(2A)$$
$$2\sin(\omega t)\cos(\omega t) = \sin(2\omega t)$$
$$\sin(\omega t)\cos(\omega t) = \frac{1}{2}\sin(2\omega t) \quad \text{[Dividing by 2]}$$

Substituting this into *:

$$p = \frac{\omega VI}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2}\sin(2\omega t)\mathrm{d}t$$
$$= \frac{\omega VI}{4\pi}\int_0^{2\pi/\omega} \sin(2\omega t)\mathrm{d}t \qquad \left[\text{Taking out } \frac{1}{2}\right]$$
$$= \frac{\omega VI}{4\pi}\left[-\frac{\cos(2\omega t)}{2\omega}\right]_0^{2\pi/\omega} \qquad \left[\text{Using} \int \sin(kt)\mathrm{d}t = -\frac{\cos(kt)}{k}\right]$$
$$= -\frac{\omega VI}{4\pi 2\omega}\left(\underbrace{\cos\left[2\omega\left(\frac{2\pi}{\omega}\right)\right]}_{\text{substituting } t=2\pi/\omega} - \underbrace{\cos(0)}_{\substack{\text{substituting}\\ t=0}}\right)$$
$$= -\frac{VI}{8\pi}\left(\underbrace{1}_{=\cos(4\pi)} - 1\right) = 0$$
$$p = 0$$

You could also have done Example 35 by using the substitution $u = \sin(\omega t)$. Try it!

There are more problems on trigonometric substitutions which are given in **Exercise 8(h).** You need to search for the appropriate trigonometric identity.

H2 **Further trigonometric substitutions**

Sometimes the substitution is not as clear-cut as in the previous examples. For some standard integrals there are suggested substitutions given in Table 4.

TABLE 4

Formulae number	*The function that needs integrating contains*	*Try*
8.54	$\sqrt{a^2 - u^2}$	$u = a\sin(\theta)$
8.55	$\sqrt{u^2 - a^2}$	$u = a\sec(\theta)$
8.56	$a^2 + u^2$	$u = a\tan(\theta)$

Let's do an example

Example 36

Show that

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

This is result 8.26 of Table 1.

Solution

Which substitution should we use?

By 8.56, let $u = a\tan(\theta)$.

We also need to replace the du. **What is du?**

Differentiating $u = a\tan(\theta)$ gives

$$\frac{du}{d\theta} \underbrace{=}_{\text{by } 6.8} a\sec^2(\theta)$$

$$du = a\sec^2(\theta)d\theta$$

By substituting these into the given integral we have

$$\int \frac{du}{a^2 + u^2} = \int \frac{a\sec^2(\theta)d\theta}{a^2 + a^2\tan^2(\theta)}$$

$$= \int \frac{a\sec^2(\theta)d\theta}{a^2\underbrace{(1 + \tan^2(\theta))}_{=\sec^2(\theta)}}$$

$$= \frac{a}{a^2}\int \frac{\sec^2(\theta)}{\sec^2(\theta)}d\theta \quad \left[\text{Taking out } \frac{a}{a^2}\right]$$

$$= \frac{1}{a}\int d\theta \quad \text{[Cancelling]}$$

Integrating dθ gives θ, hence

$$(*) \quad \int \frac{du}{a^2 + u^2} = \frac{1}{a}\theta + C$$

What is θ?

From above we have

$$a\tan(\theta) = u$$

$$\tan(\theta) = \frac{u}{a}$$

6.8 $[\tan(\theta)]' = \sec^2(\theta)$

Example 36 *continued*

?

How do we find θ?

Take inverse tan, $\tan^{-1}$, of both sides:

$$\theta = \tan^{-1}\left(\frac{u}{a}\right)$$

Replacing $\theta = \tan^{-1}\left(\frac{u}{a}\right)$ in * displays the required result:

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

Similarly we can show some of the other standard integrals of Table 1 by using the suggested substitution of Table 4.

In some cases you will be given the substitution to use, as the following example shows.

Example 37

By using the substitution $u = a\sinh(\theta)$, show that

$$\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

This is result 8.28 of Table 1.

Solution

Differentiating $u = a\sinh(\theta)$ gives

$$\frac{du}{d\theta} = a\cosh(\theta) \quad [\text{By } 6.25]$$

$$du = a\cosh(\theta)d\theta$$

Substituting $u = a\sinh(\theta)$ and $du = a\cosh(\theta)d\theta$ into the integral gives

$$\int \frac{du}{\sqrt{a^2 + u^2}} = \int \frac{a\cosh(\theta)d\theta}{\sqrt{a^2 + (a\sinh(\theta))^2}}$$

† $$= \int \frac{a\cosh(\theta)d\theta}{\sqrt{a^2 + a^2\sinh^2(\theta)}}$$

The denominator $\sqrt{a^2 + a^2\sinh^2(\theta)}$ simplifies to

$$\sqrt{a^2 + a^2\sinh^2(\theta)} = \sqrt{a^2\underbrace{[1 + \sinh^2(\theta)]}_{=\cosh^2(\theta)}} = \sqrt{a^2\cosh^2(\theta)} = a\cosh(\theta)$$

6.25 $[\sinh(\theta)]' = \cosh(\theta)$

Example 37 *continued*

Replacing this in † gives

$$\int \frac{a\cosh(\theta)\mathrm{d}\theta}{a\cosh(\theta)} = \int \mathrm{d}\theta \quad [\text{Cancelling } a\cosh(\theta)]$$

$$= \theta + C \quad [\text{Integrating}]$$

Thus we have

$$* \qquad \int \frac{\mathrm{d}u}{\sqrt{a^2 + u^2}} = \theta + C$$

We need to replace θ. From the given substitution

$$a\sinh(\theta) = u$$

we have

$$\sinh(\theta) = \frac{u}{a}$$

Take inverse sinh, $\sinh^{-1}$, of both sides:

$$\theta = \sinh^{-1}\left(\frac{u}{a}\right)$$

We have our result by substituting this into * :

$$\int \frac{\mathrm{d}u}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C$$

SUMMARY

For integrating some trigonometric functions we select a relevant trigonometric identity and then integrate.
Some of the standard integrals of Table 1 can be established by using an appropriate substitution.

Exercise **8(h)**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

Questions 1 to 5 are in the field of [*electrical principles*].

1 The average power, P, of an a.c. circuit is given by

$$P = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left(i^2 R\right)\mathrm{d}t$$

where $i = I\sin(\omega t)$, ω is angular frequency, I is peak current, R is resistance and t is time. Show that

$$P = \frac{I^2 R}{2}$$

2 The voltage, v, produced by an electronic circuit is given by

$$v = 10\sin(\omega t)$$

The root mean square value, V_{RMS}, is defined as

$$V_{RMS} = \sqrt{\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} v^2\,\mathrm{d}t}$$

Evaluate V_{RMS}.

Exercise 8(h) continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

3 The average power, P, in an a.c. circuit is given by

$$P = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}(vi)\mathrm{d}t$$

where $i = I\sin(\omega t)$ and $v = V\sin(\omega t)$.

(V and I are peak voltage and peak current respectively, ω is angular frequency and t is time.)

Show that $P = \frac{VI}{2}$.

4 The energy, W, of an inductance, L, is defined as

$$W = \frac{1}{2L}\left(\int_0^t V\mathrm{d}t + 1\right)^2$$

where V is the voltage across the inductor. For $L = 1 \times 10^{-3}$ henry and $V = \cos(t) - \sin(t)$, show that

$$W = 500[1 + \sin(2t)]$$

5 For $v = V\sin(\omega t)$ and $i = I\sin(\omega t + \phi)$, show that the power, P, in an a.c. circuit is given by

$$P = \frac{VI}{2}\cos(\phi)$$

where ϕ is the phase and P is as defined in question **3**.

6 Show that

$$\int\frac{\mathrm{d}u}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + C.$$

7 Show that $\int\frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C.$

Miscellaneous exercise 8

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

1 [*mechanics*] The kinetic energy, KE, of a body is given by

$$KE = \int_0^v (mv)\mathrm{d}v$$

where m is the mass of the body and v is the velocity. Evaluate KE.

2 [*mechanics*] The work done, W, to stretch a spring by length l is given by

$$W = \int_0^l \frac{EAx}{L}\mathrm{d}x \quad \left(\frac{EA}{L} \text{ is constant}\right)$$

where E is the modulus of elasticity, A is the cross-sectional area, L is the unstretched length and x is the displacement. Determine W.

3 [*mechanics*] The work done, W, to stretch a spring from length x_1 to length x_2 is given by

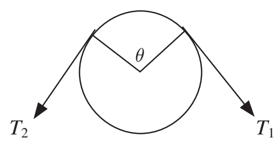
$$W = \int_{x_1}^{x_2}(kx)\mathrm{d}x$$

where k is the stiffness constant of the spring. Evaluate W.

4 [*mechanics*] Figure 6 shows a belt wound partly around a pulley. The angle θ is given by

$$\theta = \int_{T_1}^{T_2}\frac{\mathrm{d}T}{\mu T}$$

Fig. 6



where μ is the coefficient of friction between the pulley and the belt and T_1, T_2 are tensions as shown. Show that

$$T_2 = T_1 e^{\mu\theta}$$

Miscellaneous exercise **8 continued**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

5 [*structures*] The work done, W, (by external forces) on a cantilever beam of length L with uniformly distributed load q and a concentrated load p at the free end is given by

$$W = \int_0^L \frac{pq}{6EI}(2L^3 x - 3L^2 x^2 + x^4)\mathrm{d}x$$

(EI is the flexural rigidity and x is the distance along the beam).
Show that

$$W = \frac{pqL^5}{30EI}$$

6 [*materials*] The polar second moment of area, J, of a hollow circular shaft of outer diameter D_o and inner diameter D_i is given by

$$J = \int_{D_i/2}^{D_o/2} 2\pi r^3 \mathrm{d}r$$

Show that

$$J = \frac{\pi}{32}\left[D_o^4 - D_i^4\right]$$

7 Find $\displaystyle\int \frac{x^3}{x^2 - 4}\,\mathrm{d}x$.

8 [*thermodynamics*] The change in specific enthalpy, Δh, of a gas is given by

$$\Delta h = \int_{200}^{1000}\left(1.8 + (12 \times 10^{-3})T\right)\mathrm{d}T$$

Evaluate Δh.

9 [*thermodynamics*] The enthalpy change, Δh, is given by

$$\Delta h = \int_{T_1}^{T_2} C_p \mathrm{d}T$$

where $C_p = R(a + bT + cT^2 + dT^3 + eT^4)$. ($R$, a, b, c, d and e are constants.) Find an expression for Δh.

10 Find $\displaystyle\int \frac{\cos(2x)}{\cos^2(x)\sin^2(x)}\,\mathrm{d}x$.

11 [*electronics*] The potential, V, at the surface of a conductor at distance r from zero potential at distance z, is given by

$$V = -\int_z^r \frac{q}{2\pi\varepsilon_0 r}\mathrm{d}r \quad (r \neq 0)$$

where q is the charge and ε_0 is the permittivity constant. Show that

$$V = \frac{q}{2\pi\varepsilon_0}\ln\left(\frac{z}{r}\right)$$

12 [*thermodynamics*] The work done, W, by a gas is given by

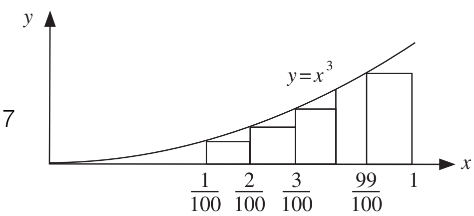
$$W = \int_{V_1}^{V_2} P\mathrm{d}V$$

where P is the pressure, V is the volume, V_1 and V_2 are the initial and final volumes of the gas respectively. If the gas obeys the law $PV = C$, where C is constant, show that $W = PV\ln\left(\frac{V_2}{V_1}\right)$.

13 [*thermodynamics*] The work done, W, on a piston by a gas in a cylinder is given by $W = \int_{V_1}^{V_2} P\mathrm{d}V$ where P is pressure and V is volume. If $PV^{1.32} = C$ (constant), find an expression for W.

14 In Fig. 7:

Fig. 7



Miscellaneous exercise **8 continued**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

i Evaluate the total area of the 100 rectangles under the graph $y = x^3$.

ii Determine $\int_0^1 x^3 \mathrm{d}x$.

iii Find the difference between the area of part **i** and $\int_0^1 x^3 \mathrm{d}x$. How can this difference be made smaller?

(*Hint*:
$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2}{4}(n+1)^2$
where n is a positive whole number.)

15 [*fluid mechanics*] The pressure, P, at a distance x of a fluid flowing around a sphere of radius r is given by

$$P = \int_{-\infty}^{x} k\left(\frac{1 + r^3/x^3}{x^4}\right)\mathrm{d}x$$

where k is a constant. Evaluate P.

16 [*structures*] A beam of length L has a uniform load w given by

$$w = w_0 \sin\left(\frac{\pi x}{L}\right)$$

where w_0 is a constant and x is the distance along the beam. The total load P and reaction R are given by

$$P = \int_0^L w\mathrm{d}x \text{ and } R = \int_0^L (wx)\,\mathrm{d}x$$

Show that **i** $P = \frac{2w_0 L}{\pi}$ and

ii $R = \frac{w_0 L^2}{\pi}$.

17 [*electronics*] In a RC (resistor capacitor) network the charging voltage w and the instantaneous voltage v are related by $\frac{t}{RC} = \int_{-w}^{v} \frac{\mathrm{d}v}{w - v}$.

Show that $v = w(1 - 2e^{-t/RC})$.

18 [*electrical principles*] The current, i, through an inductor of inductance $L = 10$ mH is given by

$$i = \frac{1}{L}\int_0^t v\mathrm{d}x$$

where v represents voltage. The voltage v across the inductor is given by

$$v = -6e^{-(2\times 10^3)x} + 10e^{-(8\times 10^3)x}$$

Determine the current i.

19 [*electrical principles*] The current, i, through an inductor of inductance L is given by

$$i = \frac{1}{L}\int_0^t v\mathrm{d}t$$

where v is the voltage across the inductor. The energy, w, of an inductance L, is defined to be

$$w = \frac{1}{2}Li^2$$

For $L = 5$ mH and $v = 10\sin(100\pi t)$, find an expression for

i i **ii** w

20 [*electromagnetism*] The flux, Φ, for a toroid of height, h, inner radius, a, and outer radius, b, is given by

$$\Phi = \int_a^b \frac{\mu i N h}{2\pi r}\mathrm{d}r \quad (r \neq 0)$$

where $a < r < b$, N is the number of turns, μ is the permeability constant and i is the current carried by the toroid coil.

If $L = N\frac{\mathrm{d}\Phi}{\mathrm{d}i}$,

show that $L = \frac{\mu h N^2}{2\pi}\ln\left(\frac{b}{a}\right)$.

21 Evaluate $\int_0^1 \frac{4}{1 + x^2}\mathrm{d}x$.

Miscellaneous exercise 8 continued

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

22 [*mechanics*] The moment of inertia, I, of a triangular lamina of height h, base b and mass m is defined as

$$I = \int_0^h \frac{2m(h-y)y^2}{h^2}\,dy$$

where y is the distance from the axis of rotation. Show that $I = \frac{mh^2}{6}$.

23 [*structures*] A cantilever beam of length L, fixed at one end and deflected by a distance D at the free end has strain energy V given by

$$V = \frac{EI}{2}\int_0^L \left(\frac{d^2 y}{dx^2}\right)^2 dx$$

where EI is the flexural rigidity. The deflection y at a distance x from the fixed end is given by

$$y = D\left[1 - \cos\left(\frac{\pi x}{2L}\right)\right]$$

Find V.

24 [*communication systems*] The total power, p, of an antenna of length L, carrying a peak current I at a distance r, is given by

$$p = \int_0^\pi \frac{\eta I^2 L^2 \pi \sin^3(\theta)d\theta}{4\lambda^2}$$

(λ = wavelength, η = space impedance and θ = angle).

Show that $p = \frac{\eta I^2 L^2 \pi}{3\lambda^2}$.

25 Sketch the area represented by $\int_1^e \frac{1}{x}\,dx$ and show that

$$\int_1^e \frac{1}{x}\,dx = 1$$

26 **i** By using a computer algebra system, or a graphical calculator, plot $y = (1 - x^2)^{1/2}$.

ii Evaluate $\int_{-1}^{1} (1 - x^2)^{1/2}dx$.

27 Show that

$$\int_0^2 (4 - x^2)^{1/2}dx = \pi$$

28 By using the substitution $u = a\cosh(\theta)$, show that

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C$$

29 By substituting $u = a\tanh(\theta)$, show that

$$\int \frac{du}{a^2 - u^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C$$

30 The Fourier coefficient, a_n, for a triangular waveform is given by

$$a_n = \frac{1}{\pi^2 n^2}\int_0^{2\pi} \omega t\cos(n\omega t)d(\omega t)$$

where n is a positive whole number. Show that $a_n = 0$.

For questions 31 and 32 use a computer algebra system or a graphical calculator.

31 [*thermodynamics*] The rate of heat flow, q, is given by

$$q = 300\int_{306}^{853} [3.7 - (1.9 \times 10^{-3})T + (4.7 \times 10^{-6})T^2 - (3.45 \times 10^{-11})T^3 + (8.5 \times 10^{-13})T^4]dT$$

Evaluate q.

Miscellaneous exercise **8 continued**

Solutions at end of book. Complete solutions available at www.palgrave.com/science/engineering/singh

32 [*thermodynamics*] Determine the specific enthalpy change, Δh, for the following:

a $$\Delta h = \int_{400}^{1000}\left[56 - \left(\frac{22 \times 10^3}{T^{0.75}}\right) + \left(\frac{116 \times 10^3}{T}\right) - \left(\frac{561 \times 10^3}{T^{1.5}}\right)\right]\mathrm{d}T$$

b $$\Delta h = \int_{400}^{1000}\left[81 - 18.66T^{0.25} + 0.54T^{0.75} - 0.04T\right]\mathrm{d}T$$

c $$\Delta h = \int_{400}^{1000}\left[143 - 58T^{0.25} + 8.3T^{0.5} - (37 \times 10^{-3})T\right]\mathrm{d}T$$

d $$\Delta h = \int_{400}^{1000}\left[(2.42 \times 10^{-6})T^2 - (41 \times 10^{-3})T + 3.05T^{0.5} - 3.7\right]\mathrm{d}T$$

(In each of these examples, T represents temperature and Δh is the specific enthalpy change of the gas being heated from 400 K to 1000 K. Also the units of Δh are kJ/kmol.)