Additional material

Chapter 2

SECTION G Applications of equations to electrical circuits

By the end of this section you will be able to:

- state Kirchhoff's laws
- apply Kirchhoff's laws to electrical circuits
- solve simultaneous equations resulting from Kirchhoff's laws
- solve simultaneous equations using a computer algebra system

G1 Modelling electrical circuits

If you are undertaking an electrical-related discipline then you may have covered Kirchhoff's laws in an electrical principles module. However, if you have not covered these laws then 2.7 and 2.8 give Kirchhoff's current and voltage laws respectively.

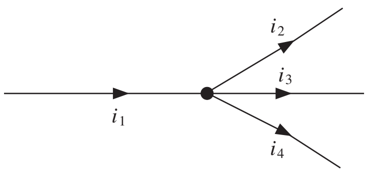
Kirchhoff's current law states

2.7 current entering a node = current leaving a node

For Fig. 43:

$$i_1 = i_2 + i_3 + i_4$$

Fig. 43



Example 20

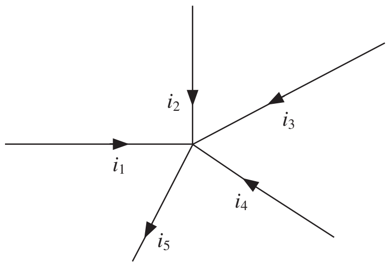
Obtain equations relating the currents i_1, i_2, i_3, i_4 and i_5 of Fig. 44.

Solution

All the currents apart from i_5 are entering the node, therefore

$$i_5 = i_1 + i_2 + i_3 + i_4$$

Fig. 44



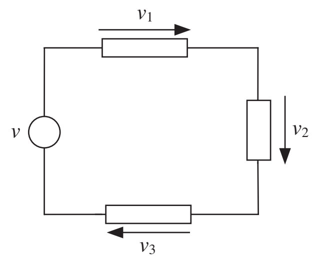
Kirchhoff's voltage law states

2.8 applied voltage = sum of the voltage drops across each component in a loop

For the circuit of Fig. 45 we have

$$v = v_1 + v_2 + v_3$$

Fig. 45

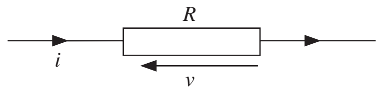


Ohm's law states that:

2.9 $v = iR$

where i is the current flowing through the resistor R and v is the voltage across the resistor R (Fig. 46).

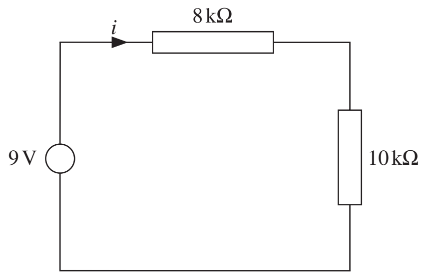
Fig. 46



Example 21

Find the current i flowing through the circuit of Fig. 47.

Fig. 47



Solution

Remember 8 kΩ is 8 kiloΩ and equals 8×10^3 Ω. By applying Kirchhoff's voltage law 2.8 to the circuit of Fig. 47 we have

$$9 = (\text{voltage drop across } 8\text{ k}\Omega) + (\text{voltage drop across } 10\text{ k}\Omega)$$

$$= i(8\text{k}) + i(10\text{k}) \qquad \text{[by applying Ohm's law]}$$

$$= i(18\text{k})$$

$$9 = (18 \times 10^3)i$$

Therefore

$$i = \frac{9}{18 \times 10^3} = 5 \times 10^{-4}$$

$$i = 0.5 \times 10^{-3} = 0.5 \text{ mA}$$

The unit mA is milliamps, 10^{-3} A.

Consider the case where we have more than one loop in the circuit. A direction (normally clockwise) for the current is chosen. The current is positive when in this direction (clockwise) and negative when in the opposite direction (anticlockwise).

Example 22

Determine the currents i_1 and i_2 in the circuit of Fig. 48.

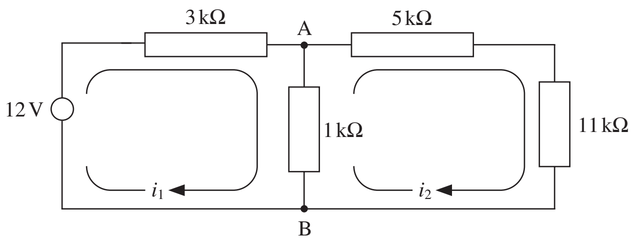


Fig. 48

Solution

What current is flowing from A to B?

$$i_1 - i_2$$

because i_1 and i_2 are going in the opposite directions. Consider the two loops. By applying Kirchhoff's voltage law 2.8 to the first loop, we have

$$12 = (\text{voltage drop across } 3\text{ k}\Omega) + (\text{voltage drop across } 1\text{ k}\Omega)$$

$$\underbrace{=}_{\text{by } 2.9} (3\text{k})i_1 + (1\text{k})(i_1 - i_2)$$

$$= (3\text{k})i_1 + (1\text{k})i_1 - (1\text{k})i_2$$

$$= (4\text{k})i_1 - (1\text{k})i_2$$

† $$12 = (4 \times 10^3)i_1 - (1 \times 10^3)i_2$$

Applying 2.8 to the second loop:

$$\underbrace{0}_{\substack{\text{no voltage source in} \\ \text{the second loop}}} = (\text{voltage drop across } 5\text{ k}\Omega) + (\text{voltage drop across } 11\text{ k}\Omega) + (\text{voltage drop across } 1\text{ k}\Omega)$$

$$0 = (5\text{k})i_2 + (11\text{k})i_2 + 1\text{k}(\underbrace{i_2 - i_1}_{\text{current from B to A}})$$

$$= (5\text{k} + 11\text{k} + 1\text{k})i_2 - (1\text{k})i_1$$

$$= (17\text{k})i_2 - (1\text{k})i_1$$

$$0 = (17 \times 10^3)i_2 - (1 \times 10^3)i_1$$

This gives

$$(1 \times 10^3)i_1 = (17 \times 10^3)i_2$$

†† $$i_1 = 17i_2 \qquad \text{[Cancelling } 10^3\text{'s]}$$

2.8 applied voltage = sum of voltage drops across each component in a loop

2.9 $v = iR$

Example 22 *continued*

We need to solve the simultaneous equations obtained:

† $\quad 12 = (4 \times 10^3)i_1 - (1 \times 10^3)i_2$

†† $\quad i_1 = 17i_2$

We can substitute $i_1 = 17i_2$ into †:

$$12 = (4 \times 10^3)17i_2 - (1 \times 10^3)i_2$$
$$= (67 \times 10^3)i_2$$

Hence

$$i_2 = \frac{12}{67 \times 10^3} = 1.79 \times 10^{-4}\ \text{A}$$

? **How can we find i_1?**

Substitute $i_2 = 1.79 \times 10^{-4}$ into $i_1 = 17i_2$:

$$i_1 = 17 \times (1.79 \times 10^{-4})$$
$$= 3.04 \times 10^{-3}\ \text{A}$$

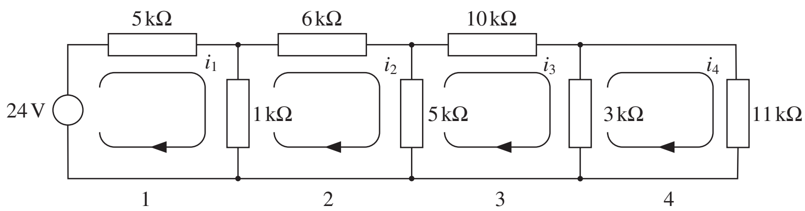
We have $i_1 = 3.04$ mA and $i_2 = 0.18$ mA.

If we consider more than two loops in a circuit then we can set up the equations using Kirchhoff's and Ohm's laws as above. However as for solving these equations, it is easier to use modern technology since it eradicates the drudgery out of the calculations. In the example below we have used a computer algebra system (MAPLE). It might be more convenient to use a graphical calculator because of its portability.

Example 23

Obtain the values of the currents i_1, i_2, i_3 and i_4 in the circuit of Fig. 49.

Fig. 49



Solution

Considering each loop separately.

Example 23 *continued*

Loop 1: Applying 2.8 we have

$$24 = (\text{voltage drop across } 5\text{ k}\Omega) + (\text{voltage drop across } 1\text{ k}\Omega)$$
$$= (5\text{k})i_1 + (1\text{k})(i_1 - i_2) \qquad [\text{by } 2.9]$$
$$= (6\text{k})i_1 - (1\text{k})i_2$$
$$24 = (6 \times 10^3)i_1 - (1 \times 10^3)i_2$$

Loop 2: Similarly

$$0 = (\text{voltage drop across } 6\text{ k}\Omega) + (\text{voltage drop across } 5\text{ k}\Omega) + (\text{voltage drop across } 1\text{ k}\Omega)$$
$$= (6\text{k})i_2 + (5\text{k})(i_2 - i_3) + (1\text{k})(i_2 - i_1)$$
$$= (12\text{k})i_2 - (1\text{k})i_1 - (5\text{k})i_3$$
$$0 = -(1 \times 10^3)i_1 + (12 \times 10^3)i_2 - (5 \times 10^3)i_3$$

Loop 3: We have

$$0 = (\text{voltage drop across } 10\text{ k}\Omega) + (\text{voltage drop across } 3\text{ k}\Omega) + (\text{voltage drop across } 5\text{ k}\Omega)$$
$$= (10\text{k})i_3 + (3\text{k})(i_3 - i_4) + (5\text{k})(i_3 - i_2)$$
$$= (18\text{k})i_3 - (3\text{k})i_4 - (5\text{k})i_2$$
$$0 = -(5 \times 10^3)i_2 + (18 \times 10^3)i_3 - (3 \times 10^3)i_4$$

Loop 4:

$$0 = (\text{voltage drop across } 11\text{ k}\Omega) + (\text{voltage drop across } 3\text{ k}\Omega)$$
$$= (11\text{k})i_4 + (3\text{k})(i_4 - i_3)$$
$$= (14\text{k})i_4 - (3\text{k})i_3$$
$$0 = -(3 \times 10^3)i_3 + (14 \times 10^3)i_4$$

Combining these four equations gives

$$(6 \times 10^3)i_1 - (1 \times 10^3)i_2 = 24$$
$$-(1 \times 10^3)i_1 + (12 \times 10^3)i_2 - (5 \times 10^3)i_3 = 0$$
$$-(5 \times 10^3)i_2 + (18 \times 10^3)i_3 - (3 \times 10^3)i_4 = 0$$
$$-(3 \times 10^3)i_3 + (14 \times 10^3)i_4 = 0$$

2.8 applied voltage = sum of voltage drops across each component in a loop
2.9 $v = iR$

Example 23 *continued*

Solving these using MAPLE we obtain (*10^3 can also be replaced by e3 in MAPLE)

```
> eqn1:=(6*10^3)*i[1] - (1*10^3)*i[2] = 24;
```

$$\text{eqn }1 := 6000\, i_1 - 1000\, i_2 = 24$$

```
> eqn2:= - (1*10^3)*i[1] + (12*10^3)*i[2] - (5*10^3)*i[3] = 0;
```

$$\text{eqn }2 := -1000\, i_1 + 12000\, i_2 - 5000\, i_3 = 0$$

```
> eqn3:= - (5*10^3)*i[2] + (18*10^3)*i[3] - (3*10^3)*i[4] = 0;
```

$$\text{eqn }3 := -5000\, i_2 + 18000\, i_3 - 3000\, i_4 = 0$$

```
> eqn4:= - (3*10^3)*i[3] + (14*10^3)*i[4] = 0;
```

$$\text{eqn }4 := -3000\, i_3 + 14000\, i_4 = 0$$

```
> evalf (solve ({eqn1, eqn2, eqn3, eqn4}));
```

$$\{i_1 = .004064145714,\ i_3 = .0001108691348,\ i_4 = .00002375767175,\ i_2 = .0003848742823\}$$

Rounding to 2 d.p. we have

$$i_1 = 4.06 \text{ mA},\ i_2 = 0.38 \text{ mA},\ i_3 = 0.11 \text{ mA and } i_4 = 0.02 \text{ mA}$$

SUMMARY

Kirchhoff's current law:

2.7 current entering a node = current leaving a node

Kirchoff's voltage law:

2.8 applied voltage = sum of voltage drops across each component in a loop

Ohm's law:

2.9 $v = iR$

Exercise **2(g)**

Solutions are given at the end of this additional material.
Complete solutions are in this website.

1 Ascertain an expression relating the currents i, i_1, i_2 and i_3 for the circuit of Fig. 50.

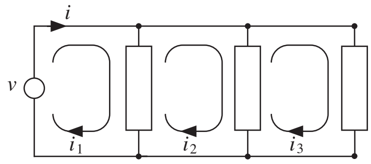


Fig. 50

2 Write an expression relating the currents i_1, i_2, i_3, i_4 and i_5 for the circuit of Fig. 51.

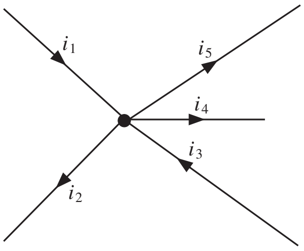


Fig. 51

3 Obtain four relationships, one for each node A, B, C, D, between the currents i_1, i_2, i_3, i_4 and i_5 for the circuit of Fig. 52.

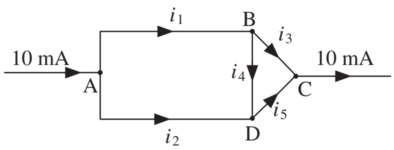


Fig. 52

4 Find the current i flowing through the circuit of Fig. 53.

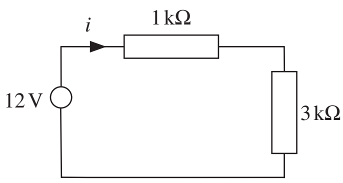


Fig. 53

5 Obtain a value for the current i of the circuit of Fig. 54.

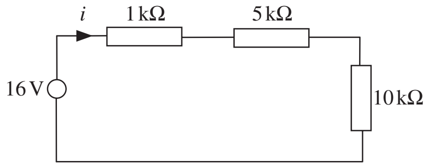


Fig. 54

6 Obtain the values of the currents i_1 and i_2 for the circuit of Fig. 55.

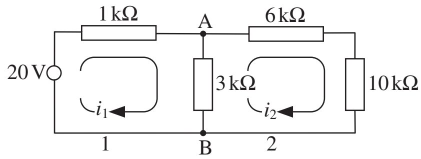


Fig. 55

7 Find the currents i_1, i_2 and i_3 shown in the circuit of Fig. 56.

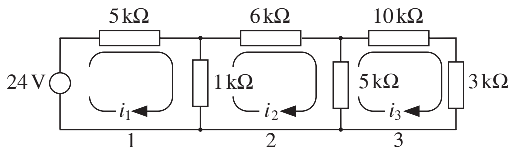


Fig. 56

8 Establish the values of the currents i_1, i_2 and i_3 for the circuit of Fig. 57.

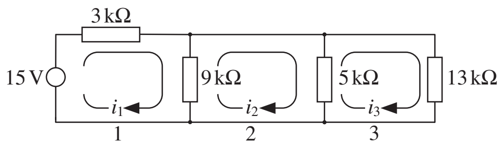


Fig. 57

9 Find the values of the currents i_1, i_2, i_3 and i_4 for the circuit of Fig. 58.

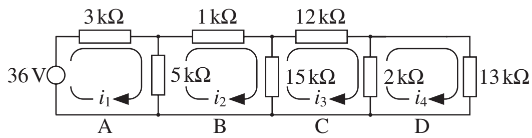


Fig. 58

10 Determine the values of currents i_1, i_2 and i_3 for the circuit of Fig. 59.

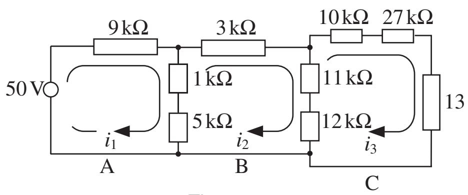


Fig. 59

Chapter 3

G Step function

In electronics engineering, one of the most important functions is the step function, denoted $H(t)$, which is defined as

3.8 $$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

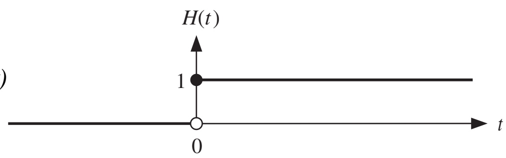
$H(t)$ is a function that depends on time, t, and has a value of zero for $t < 0$ and one for $t \geq 0$.

> **Why is the step function denoted by $H(t)$?**
> It was **Oliver Heaviside** (1850–1925) who developed these step functions, hence $H(t)$. He was born in Camden Town, London and at a young age became deaf. However he was interested in academic subjects but detested the rigour of mathematics and chose to publish papers in electromagnetism. In 1891 he was elected a Fellow of the prestigious Royal Society. So the H in the step function refers to Oliver Heaviside. Step function is also known as 'Heaviside' function.

? What does the graph of this function look like?

The graph of $H(t)$ is shown in Fig. 28.

Fig. 28
Graph of $H(t)$



The function jumps at $t = 0$ and has a value of 1 at this point and for $t > 0$.

$H(t)$ is sometimes called the 'switch' function (it switches on at $t = 0$).

? What do ● and ○ signify in the graph of Fig. 28?

The points ● and ○ at $t = 0$ represent the fact that $H(t)$ has a value of 1 at this point and **not** zero.

There is also the delayed step function, $H(t - a)$, given by

3.9 $$H(t-a) = \begin{cases} 1 & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

This $H(t - a)$ switches on at $t = a$. The graph has the shape shown in Fig. 29.

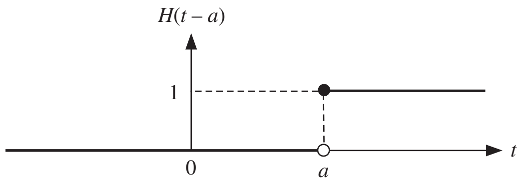


Fig. 29 Graph of $H(t - a)$

Notice that the graph jumps at $t = a$ to a value of 1 for the delayed step function.

Step functions may have other values besides 1. For example, the graph of $bH(t - a)$ has the shape shown in Fig. 30.

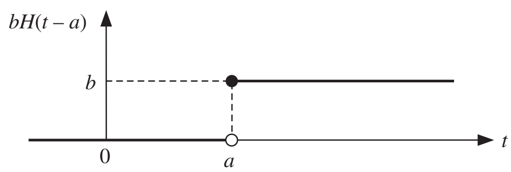


Fig. 30 Graph of $bH(t - a)$

The graph hops from zero to a value of b at a and then stays at this value. It is defined by

3.10 $$bH(t-a) = \begin{cases} b & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

Let's try some examples. From now on we will not always plot graphs with ● and ○ and will assume that the graph follows the definition given above in 3.8, 3.9 and 3.10.

Example 25 *electronics*

The voltage, $v(t)$, applied to a circuit is given by

a $v(t) = H(t - 3)$ **b** $v(t) = 5H(t - 3)$ **c** $v(t) = 5H(t - 1) - 5H(t - 2)$

Sketch these functions on different axes.

Solution

a The graph of $v(t) = H(t - 3)$ means the graph switches on at $t = 3$ and has a value of 1. More rigorously, we can use 3.9 with $a = 3$ which gives

$$H(t-3) = \begin{cases} 1 & \text{if } t \geq 3 \\ 0 & \text{if } t < 3 \end{cases}$$

and the graph has the shape shown in Fig. 31.

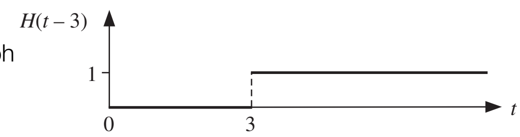


Fig. 31 Graph of $H(t - 3)$

b The graph $v(t) = 5H(t - 3)$ switches on at $t = 3$ and has a value of 5. Putting $b = 5$ and $a = 3$ into 3.10 gives

$$5H(t-3) = \begin{cases} 5 & \text{if } t \geq 3 \\ 0 & \text{if } t < 3 \end{cases}$$

(see Fig. 32).

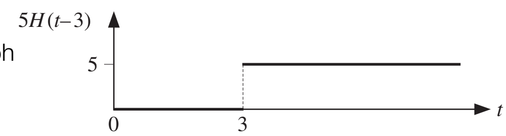


Fig. 32 Graph of $5H(t - 3)$

c For $v(t) = 5H(t - 1) - 5H(t - 2)$, we can consider each part by using 3.10:

$$5H(t-1) = \begin{cases} 5 & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases} \quad \text{and} \quad 5H(t-2) = \begin{cases} 5 & \text{if } t \geq 2 \\ 0 & \text{if } t < 2 \end{cases}$$

? **For $t < 1$, $5H(t - 1) = 0$ and $5H(t - 2) = 0$, so $v(t) = ?$**

$$v(t) = 0 - 0 = 0$$

? **For $1 \leq t < 2$, $5H(t - 1) = 5$ and $5H(t - 2) = 0$, so $v(t) = ?$**

$$v(t) = 5 - 0 = 5$$

For $t \geq 2$, $5H(t - 1) = 5$ and $5H(t - 2) = 5$, so $v(t) = 5 - 5 = 0$.

3.10 $$bH(t-a) = \begin{cases} b & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

Example 25 *continued*

Combining these three pieces we have the graph shown in Fig. 33.

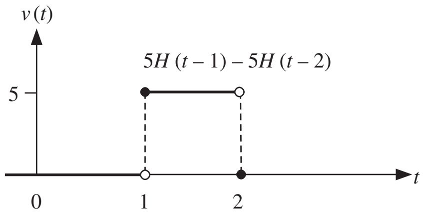


Fig. 33 Graph of $5H(t-1)-5H(t-2)$

Observe that the graph of $v(t) = 5H(t-1) - 5H(t-2)$ is a pulse of value 5 between 1 and 2. Also at $t = 1$, $v(t) = 5$ and at $t = 2$, $v(t) = 0$. A function of this format, $5H(t-1) - 5H(t-2)$, will always be a pulse. For example, the general function $f(t) = H(t-a) - H(t-b)$ has a pulse of height 1 between $t = a$ and $t = b$, and zero elsewhere.

Step functions can also take up other values besides constants. For example, the graph of $g(t)H(t-a)$ has the shape shown in Fig. 34.

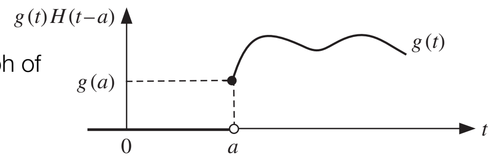


Fig. 34 Graph of $g(t)H(t-a)$

The graph hops from zero to a value of $g(a)$ at a and then traces the graph of $g(t)$ for $t \geq a$. It is defined by

3.11 $$g(t)H(t-a) = \begin{cases} g(t) & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

Example 26 *electronics*

The input voltage, $v(t)$, to an amplifier is given by

$$v(t) = t^2H(t-2)$$

Sketch this function.

Solution

For $v(t) = t^2H(t-2)$, putting $a = 2$ and $g(t) = t^2$ into

3.11 $$g(t)H(t-a) = \begin{cases} g(t) & \text{if } t \geq a \\ 0 & \text{if } t < a \end{cases}$$

Example 26 *continued*

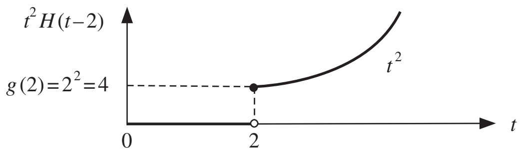
gives

$$t^2H(t-2) = \begin{cases} t^2 & \text{if } t \geq 2 \\ 0 & \text{if } t < 2 \end{cases}$$

How do we sketch this graph?

Well, $v(t)$ switches on at $t = 2$ and then it traces the graph of t^2 from 2 onwards (Fig. 35).

Fig. 35 Graph of $t^2H(t-2)$



SUMMARY

The basic step function is defined by

3.8 $$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The step function is sometimes called the 'switch' function.

Exercise 3(g)

Solutions are given at the end of this additional material. Complete solutions are in this website.

1 Sketch the following functions:

a $f(t) = H(t-2)$

b $f(t) = 2H(t-2)$

c $f(t) = tH(t-1)$

d $f(t) = (t^2 - 2t + 1)H(t-1)$

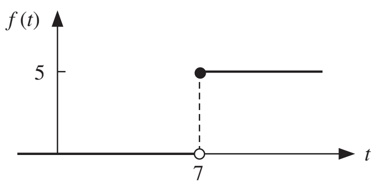
e $f(t) = H(t-2) - H(t-3)$

f $f(t) = t^2[H(t-2) - H(t-3)]$

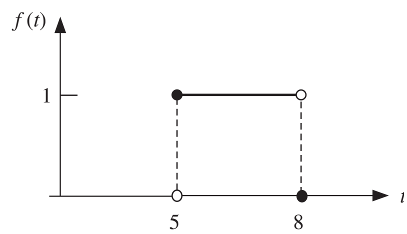
g $f(t) = 5H(t-2) - 6H(t-3)$

2 Write expressions for the functions, $f(t)$, of the following graphs:

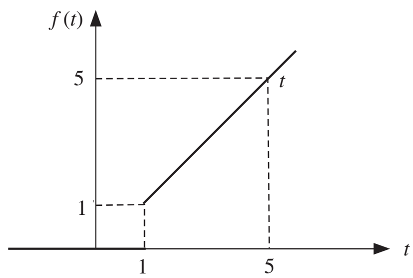
a



b



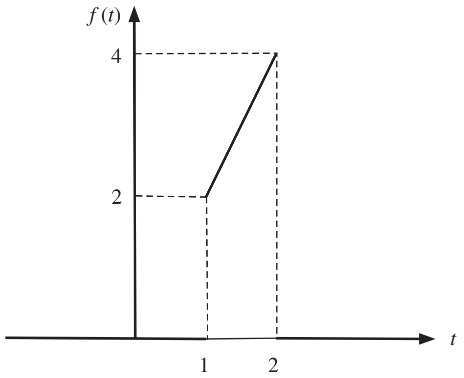
c



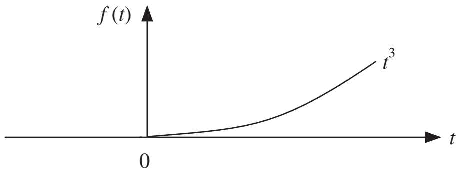
Exercise 3(g)

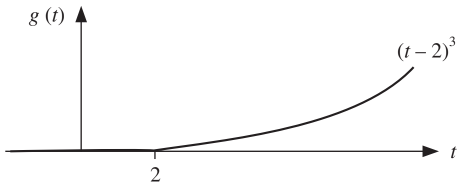
Solutions are given at the end of this additional material. Complete solutions are in this website.

d



3 **i** Write the functions, $f(t)$ and $g(t)$, for the following graphs:





ii Sketch the graph of $f(t) = (t - 5)^3 H(t - 5)$

iii Sketch the graph of $f(t) = (t^3 - 15t^2 + 15t - 125)\, H(t - 5)$

Miscellaneous exercise 3 (extra)

Solutions are given at the end of this additional material. Complete solutions are in this website.

19 Sketch the following functions:

a $f(t) = H(t - 3)$

b $f(t) = 5H(t - 3) - 5H(t - 4)$

c $f(t) = (t^2 - 4t + 4)H(t - 2)$

Chapter 5

SECTION F **Hyperbolic properties**

By the end of this section you will be able to:

- evaluate other hyperbolic functions
- show hyperbolic identities
- understand inverse hyperbolic functions

F1 Other hyperbolic functions

We define hyperbolic functions – cosech, sech and coth – in a similar way to the definitions of trigonometric functions cosec, sec and cot respectively:

5.33 $$\operatorname{cosech}(x) = \frac{1}{\sinh(x)} \quad [\sinh(x) \neq 0]$$

5.34 $$\operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

5.35 $$\coth(x) = \frac{1}{\tanh(x)} = \frac{\cosh(x)}{\sinh(x)} \quad [\sinh(x) \neq 0]$$

Note the similarity with the analogous trigonometric definition:

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)}$$

We use a calculator to evaluate these functions.

Example 23

Determine cosech(0.3), sech(5) and coth(5000).

Solution

By 5.33 we have

$$\operatorname{cosech}(0.3) = \frac{1}{\sinh(0.3)} = [\sinh(0.3)]^{-1}$$

For cosech(0.3), we evaluate $[\sinh(0.3)]^{-1}$ on a calculator. PRESS

(**hyp** **sin** **0.3** **)** **x^{-1}** **=** which should show 3.283853397.

By using a calculator we have sech(5) = 0.013 and coth (5000) = 1.

F2 **Hyperbolic identities**

Example 24

Show that

5.36 $\coth^2(x) - 1 = \text{cosech}^2(x)$

Solution

We use the fundamental identity,

5.32 $\cosh^2(x) - \sinh^2(x) = 1$

Dividing both sides of this identity by $\sinh^2(x)$ gives

$$\frac{\cosh^2(x)}{\sinh^2(x)} - \frac{\sinh^2(x)}{\sinh^2(x)} = \frac{1}{\sinh^2(x)}$$

$$\frac{\cosh^2(x)}{\sinh^2(x)} - 1 = \frac{1}{\sinh^2(x)}$$

$$\coth^2(x) - 1 = \text{cosech}^2(x)$$

The last line follows by using $\frac{\cosh(x)}{\sinh(x)} = \coth(x)$ and $\frac{1}{\sinh(x)} = \text{cosech}(x)$.

We can use different variables after the hyperbolic function, it doesn't need to be x. For example $\sinh(A)$.

Note the similarity in the identities of the hyperbolic and trigonometric functions in Table 13.

TABLE 13

Trigonometric	*Hyperbolic*
$\cos^2(A) + \sin^2(A) = 1$	$\cosh^2(A) - \sinh^2(A) = 1$
$\cot^2(A) + 1 = \text{cosec}^2(A)$	$\coth^2(A) - 1 = \text{cosech}^2(A)$
$1 + \tan^2(A) = \sec^2(A)$	$1 - \tanh^2(A) = \text{sech}^2(A)$

There is a technique to move from the trigonometric identity to the analogous hyperbolic identity. We use **Osborne's rule** which says that the trigonometric identity can be replaced by the analogous hyperbolic identity but the sign of any direct (or implied) product of two sinh's must be changed.

For example in trigonometry we have $\cos^2(A) + \sin^2(A) = 1$. Applying Osborne's rule:

$$\cosh^2(A) - \underbrace{\sinh^2(A)}_{\text{direct product of two sinh's}} = 1$$

Remember that $\sinh^2(A) = \sinh(A) \times \sinh(A)$ – so the positive sign (+) in the middle changes to a negative (−) sign.

Similarly in trigonometry: $\cot^2(A) + 1 = \text{cosec}^2(A)$. Using Osborne's rule we have

* $$-\coth^2(A) + 1 = -\text{cosech}^2(A)$$

because $\coth^2(A) = \dfrac{\cosh^2(A)}{\sinh^2(A)}$ and $\text{cosech}^2(A) = \dfrac{1}{\sinh^2(A)}$ – in both cases there is an implied product of two sinh's.

Multiplying both sides of * by −1 gives

$$\coth^2(A) - 1 = \text{cosech}^2(A)$$

This identity is also verified above in **Example 24**.

There are many other hyperbolic identities which can be shown by Osborne's rule.
Try verifying some of the following identities:

5.37 $$1 - \tanh^2(A) = \text{sech}^2(A)$$

5.38 $$\cosh(2A) = \cosh^2(A) + \sinh^2(A) = 2\cosh^2(A) - 1 = 1 + 2\sinh^2(A)$$

5.39 $$\sinh(2A) = 2\sinh(A)\cosh(A)$$

5.40 $$\tanh(2A) = \frac{2\tanh(A)}{1 + \tanh^2(A)}$$

5.41 $$\sinh(A \pm B) = \sinh(A)\cosh(B) \pm \cosh(A)\sinh(B)$$

5.42 $$\cosh(A \pm B) = \cosh(A)\cosh(B) \pm \sinh(A)\sinh(B)$$

5.43 $$\tanh(A \pm B) = \frac{\tanh(A) \pm \tanh(B)}{1 \pm \tanh(A)\tanh(B)}$$

5.44 $$\sinh(A) + \sinh(B) = 2\sinh\left(\frac{A + B}{2}\right)\cosh\left(\frac{A - B}{2}\right)$$

5.45 $$\sinh(A) - \sinh(B) = 2\cosh\left(\frac{A + B}{2}\right)\sinh\left(\frac{A - B}{2}\right)$$

5.46 $$\cosh(A) + \cosh(B) = 2\cosh\left(\frac{A + B}{2}\right)\cosh\left(\frac{A - B}{2}\right)$$

5.47 $$\cosh(A) - \cosh(B) = 2\sinh\left(\frac{A + B}{2}\right)\sinh\left(\frac{A - B}{2}\right)$$

For example, to show 5.41 :

$$\sinh(A + B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$$

Notice that there is **no** direct or implied product of two sinh's, thus the hyperbolic identity is the same as the trigonometric identity:

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

By Osborne's rule:

$$\sinh(A + B) = \sinh(A)\cosh(B) + \cosh(A)\sinh(B)$$

F3 Inverse hyperbolic functions

The inverse hyperbolic functions of $\sinh(x)$, $\cosh(x)$ and $\tanh(x)$ are denoted by $\sinh^{-1}(x)$, $\cosh^{-1}(x)$ and $\tanh^{-1}(x)$ respectively.

These functions are sometimes designated by arsinh, arcosh and artanh.

? **What does $\sinh^{-1}$ represent?**

If $\sinh(y) = x$ then

$$y = \sinh^{-1}(x)$$

(The following are correct to 3 d.p.) For example, $\sinh(2.1) = 4.022$ therefore

$$\sinh^{-1}(4.022) = 2.1$$

? **What is $\sinh^{-1}(3.627)$, given that $\sinh(2) = 3.627$?**

$$\sinh^{-1}(3.627) = 2$$

Similarly if $\cosh(y) = x$ then

$$y = \cosh^{-1}(x) \qquad (x \geq 1)$$

The domain of inverse cosh function is $x \geq 1$.

? **What is $\cosh^{-1}(1)$ equal to, given that $\cosh(0) = 1$?**

$$\cosh^{-1}(1) = 0$$

From $\tanh(y) = x$ it follows that

$$y = \tanh^{-1}(x) \qquad (-1 < x < 1)$$

The domain of the inverse tanh lies between -1 and $+1$, that is $-1 < x < 1$.

To evaluate these inverse functions we can use a calculator.

Example 25

Determine, correct to three d.p., $\sinh^{-1}(3)$, $\sinh^{-1}(-3)$, $\cosh^{-1}(3)$, $\tanh^{-1}(0)$, $\tanh^{-1}(0.25)$ and $\tanh^{-1}(1)$.

Solution

Using a calculator to evaluate $\sinh^{-1}(3)$, PRESS **hyp** **SHIFT** **sin** **3** **=** which should show 1.818446459.

So $\sinh^{-1}(3) = 1.818$. Similarly we have:

? $\sinh^{-1}(-3) = -1.818$, $\cosh^{-1}(3) = 1.763$, $\tanh^{-1}(0) = 0$, $\tanh^{-1}(0.25) = 0.255$ and for $\tanh^{-1}(1)$, the calculator shows an error. **Why?**

The function $\tanh^{-1}(x)$ is only valid for x between -1 and $+1$ and is not a real number for $x \geq 1$ or $x \leq -1$. (See Fig. 14**c** below.)

You can plot the inverse hyperbolic functions on a graphical calculator or a computer algebra system (Fig. 14).

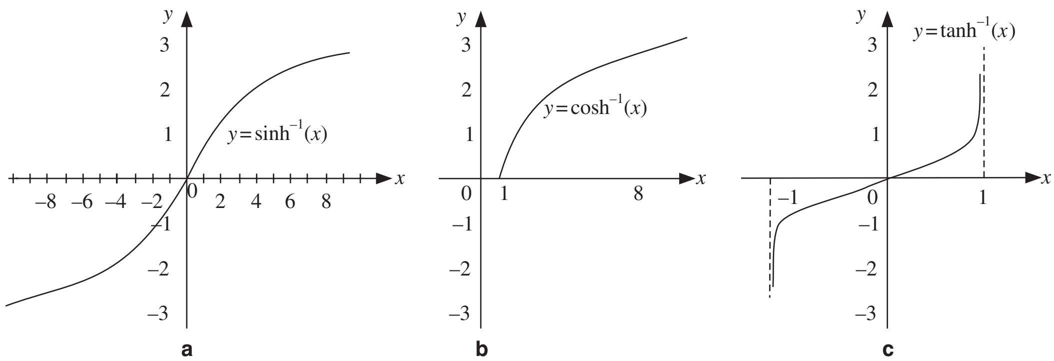


Fig. 14

? **Do you notice why we cannot evaluate** $\tanh^{-1}(1)$**?**

There is a vertical asymptote at $x = 1$. Similarly we cannot evaluate $\tanh^{-1}(-1)$.

As can be seen by the graph of Fig. 14**b**, the inverse cosh function, $\cosh^{-1}$, is only valid for x greater than or equal to 1. If we try to evaluate $\cosh^{-1}(x)$ for x values less than 1, the calculator shows an error.

Example 26 *mechanics*

The length, s, of a cable can be found from

$$* \qquad x = \frac{T}{w}\sinh^{-1}\left(\frac{sw}{T}\right)$$

where T is tension, w is load per unit length and x is horizontal distance. Show that

$$s = \frac{T}{w}\sinh\left(\frac{wx}{T}\right)$$

Solution

Multiplying both sides of the given equation, *, by w gives

$$wx = T\sinh^{-1}\left(\frac{sw}{T}\right)$$

We need to obtain s from the Right-Hand Side. Divide both sides by T:

$$\frac{wx}{T} = \sinh^{-1}\left(\frac{sw}{T}\right)$$

How do we remove $\sinh^{-1}$?

Take sinh of both sides:

$$\sinh\left(\frac{wx}{T}\right) = \sinh\left[\sinh^{-1}\left(\frac{sw}{T}\right)\right] = \frac{sw}{T}$$

(because $\sinh^{-1}$ is the inverse function of sinh).

Transposing to make s the subject gives $s = \frac{T}{w}\sinh\left(\frac{wx}{T}\right)$.

SUMMARY

The hyperbolic identities can be established from the analogous trigonometric identities by using Osborne's rule which says that the sign of the product of two sinh's must be changed.

Inverse hyperbolic functions are denoted by $\sinh^{-1}$, $\cosh^{-1}$ and $\tanh^{-1}$. We can evaluate these functions on a calculator.

Exercise 5(f)

Solutions are given at the end of this additional material. Complete solutions are in this website.

1 Evaluate sech(2), cosech(2) and coth(10).

2 Find $\sinh^{-1}(\pi)$, $\sinh^{-1}(-\pi)$, $\tanh^{-1}(0)$, $\tanh^{-1}(0.5)$, $\cosh^{-1}(\pi)$, $\cosh^{-1}(1000)$ and $\cosh^{-1}(0)$.

3 Without using a calculator, determine

$\sinh[\sinh^{-1}(\pi)]$, $\sinh[\sinh^{-1}(5)]$,

$\cosh[\cosh^{-1}(\pi)]$ and $\tanh[\tanh^{-1}(0.236)]$

4 Find x which satisfies

a $\cosh(x) = 1.7$

b $\sinh(x) = \pi$

c $\tanh(x) = 0.5$

5 Without using Osborne's rule, show that

a $1 - \tanh^2(x) = \text{sech}^2(x)$

b $2\sinh(x)\cosh(x) = \sinh(2x)$

Use a computer algebra system or a graphical calculator for question 6.

6 Plot on different axes the following graphs for x between -10 and 10:

a $y = \text{sech}(x)$

b $y = \coth(x)$

c $y = \text{cosech}(x)$

7 Show that

$\cosh(2x) = \cosh^2(x) + \sinh^2(x)$

8 Show that

$$\cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 = 1 + 2\sinh^2(x)$$

9 Without using Osborne's rule, show that

$$2\sinh\left(\frac{A+B}{2}\right)\cosh\left(\frac{A-B}{2}\right) = \sinh(A) + \sinh(B)$$

10 *[mechanics]* The length, s, of a cable with span L and sag h can be determined by

$$s = \frac{L}{2}\left\{\left[1 + \left(\frac{4h}{L}\right)^2\right]^{1/2} + \left(\frac{L}{4h}\right)\sinh^{-1}\left(\frac{4h}{L}\right)\right\}$$

Find the length of the cable which has a span of 200 m and a sag of 60 m.

11 *[mechanics]* The length, s, of a cable can be evaluated from the equation

$$x = \frac{T}{w}\sinh^{-1}\left[\frac{ws}{T} + \tan^{-1}(\theta)\right]$$

where T represents horizontal tension, w is load per unit length, $\tan^{-1}(\theta)$ is an angle and x is horizontal distance. Make s the subject of the equation.

12 *[electrical principles]* A transmission line of length L has an impedance Z given by

$$Z = \frac{2Z_0e^{-\gamma L}}{(1 + e^{-\gamma L})(1 - e^{-\gamma L})}$$

where Z_0 is the characteristic impedance and γ is the propagation coefficient. Show that

$$Z = Z_0\,\text{cosech}(\gamma L)$$

Miscellaneous exercise 5 (extra)

Solutions are given at the end of this additional material. Complete solutions are in this website.

16 Without using Osborne's rule, show that

$$\tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x)\tanh(y)}$$

17 [*electrical principles*] In a symmetrical network we have the following equations:

$$Z_1 + Z_2 = 2Z_0\coth(\gamma L) \quad (*)$$

$$\frac{2Z_1Z_2}{Z_1 + Z_2} = Z_0\tanh(\gamma L) \quad (**)$$

Show that

$$Z_2 = Z_0[\coth(\gamma L) \pm \operatorname{cosec}(\gamma L)]$$

(Z_0, Z_1 and Z_2 are impedences, γ is the propagation coefficient and L is the length.)

For question 18 use a computer algebra system (or a graphical calculator).

18 [*mechanics*] The length, s, of a cable with span L and sag h is given by

$$s = \frac{L}{2}\left\{\left[1 + \left(\frac{4h}{L}\right)^2\right]^{1/2} + \left(\frac{L}{4h}\right)\sinh^{-1}\left(\frac{4h}{L}\right)\right\}$$

a Plot the graph of s for $-60 \le h \le 0$ with $L = 200$ m.

b Determine h, if $L = 200$ m and $s = 240.87$ m.

19 [*electrical principles*] A transmission line of length L has a sending end voltage V_s and sending end current I_s given by

$$V_s = V\cosh(\gamma L) + IZ\sinh(\gamma L) \quad (\dagger)$$

$$I_s = I\cosh(\gamma L) + \frac{V}{Z}\sinh(\gamma L) \quad (\dagger\dagger)$$

where V is receiving end voltage, I is receiving end current, Z is characteristic impedance and γ is propagation coefficient. Show that

$$I = I_s\cosh(\gamma L) - \frac{V_s}{Z}\sinh(\gamma L)$$

$$V = V_s\cosh(\gamma L) - ZI_s\sinh(\gamma L)$$

Chapter 10

SECTION F **Functions of complex numbers**

By the end of this section you will be able to:

- use some identities between trigonometric and hyperbolic functions
- establish some of these identities
- apply these to engineering examples

This section is a lot **more** difficult than previous sections. In this section we establish and state a number of identities involving complex numbers.

F1 **Identities**

In this section we use the fundamental identities derived in the previous section.

10.25 $$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

10.26 $$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

Remember that θ needs to be in radians.

Example 29

Show that

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \cos(\theta)$$

Solution

Expanding the Left-Hand Side:

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{\overbrace{\cos(\theta) + j\sin(\theta)}^{\text{by } 10.25} + \overbrace{\cos(\theta) - j\sin(\theta)}^{\text{by } 10.26}}{2}$$

$$= \frac{2\cos(\theta)}{2} = \cos(\theta) \quad \text{[Cancelling 2's]}$$

We define the complex trigonometric functions $\sin(z)$ and $\cos(z)$ as follows:

10.27 $$\cos(z) = \frac{e^{jz} + e^{-jz}}{2} \quad \text{[Replace } \theta \text{ by } z \text{ in } \textbf{Example 29}]$$

Similarly

10.28 $$\sin(z) = \frac{e^{jz} - e^{-jz}}{2j}$$

From these we can obtain

10.29 $$\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{1}{j}\left[\frac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}}\right]$$

We define the complex hyperbolic functions as

10.30 $$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

10.31 $$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

10.32 $$\tanh(z) = \frac{\sinh(z)}{\cosh(z)} = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

Example 30

Show that

$$\sin(jz) = j\sinh(z)$$

Solution

By

10.28 $$\sin(z) = (e^{jz} - e^{-jz})/2j$$

$$\sin(jz) = \frac{e^{j(jz)} - e^{-j(jz)}}{2j}$$

$$= \frac{e^{j^2 z} - e^{-j^2 z}}{2j}$$

$$= \frac{e^{-z} - e^{z}}{2j} \quad \text{[Because } j^2 = -1\text{]}$$

$$\underbrace{=}_{\text{by } 10.13} \frac{-j2(e^{-z} - e^{z})}{4} \quad \text{[Complex conjugate of } j2 \text{ is } -j2\text{]}$$

$$= -j\left(\frac{e^{-z} - e^{z}}{2}\right) \quad \left[\text{Because } \frac{2}{4} = \frac{1}{2}\right]$$

$$= j\left(\frac{e^{z} - e^{-z}}{2}\right) = j\underbrace{\sinh(z)}_{\text{by } 10.31}$$

10.13 $\dfrac{a + jb}{c + jd} = \dfrac{(a + jb)(c - jd)}{c^2 + d^2}$ 10.31 $\sinh(z) = (e^{z} - e^{-z})/2$

Similarly we have the following identities:

10.33 $\cos(jz) = \cosh(z)$

10.34 $\sin(jz) = j\sinh(z)$

10.35 $\tan(jz) = j\tanh(z)$

10.36 $\cosh(jz) = \cos(z)$

10.37 $\sinh(jz) = j\sin(z)$

10.38 $\tanh(jz) = j\tan(z)$

You are asked to show some of these identities in **Exercise 10(f)**. Many of the properties of real trigonometric functions also apply to complex trigonometric functions. We will not list them here but just apply them in the appropriate case, as the following example shows.

Example 31

Determine x and y given that

$$x + jy = \cos(0.12 + j3)$$

(x and y are real.)

Solution

We use

4.39 $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$

$$\cos(0.12 + j3) = \cos(0.12)\cos(j3) - \sin(0.12)\sin(j3)$$

* $$= 0.99\cos(j3) - 0.12\sin(j3)$$

? **What is cos(j3) and sin(j3) equal to?**

$$\cos(j3) \underbrace{=}_{\text{by } 10.33} \cosh(3) = 10.07 \quad \text{[Via calculator]}$$

$$\sin(j3) \underbrace{=}_{\text{by } 10.34} j\sinh(3) = j10.02 \quad \text{[Via calculator]}$$

Substituting these values into * gives

$$\cos(0.12 + j3) = (0.99 \times 10.07) - (0.12 \times j10.02)$$
$$= 9.97 - j1.20$$

Equating the real and imaginary parts of $x + jy = 9.97 - j1.20$ gives

$$x = 9.97 \text{ and } y = -1.20$$

10.33 $\cos(jz) = \cosh(z)$ 10.34 $\sin(jz) = j\sinh(z)$

The next example might seem like a colossal jump from previous examples. Don't be put off by all the different symbols used in the example, we still use the same rules of complex numbers.

Example 32 *electrical principles*

A transmission line of length L with a characteristic impedance z_0 has an input impedance z_{input} given by

$$\dagger \qquad z_{\text{input}} = z_0\left[\frac{z_L \cosh(\gamma L) + z_0 \sinh(\gamma L)}{z_0 \cosh(\gamma L) + z_L \sinh(\gamma L)}\right]$$

where γ = propagation coefficient and z_L = load. Show that if $\gamma = j\beta$ then

$$z_{\text{input}} = z_0\left[\frac{z_L + jz_0\tan(\beta L)}{z_0 + jz_L\tan(\beta L)}\right]$$

Solution

Dividing the numerator and denominator of † by $\cosh(\gamma L)$:

$$z_{\text{input}} = z_0\left[\frac{z_L + z_0\dfrac{\sinh(\gamma L)}{\cosh(\gamma L)}}{z_0 + z_L\dfrac{\sinh(\gamma L)}{\cosh(\gamma L)}}\right] \qquad \left[\text{Remember } \frac{\cosh(\gamma L)}{\cosh(\gamma L)} = 1\right]$$

$$\underbrace{=}_{\text{by } 5.27} z_0\left[\frac{z_L + z_0\tanh(\gamma L)}{z_0 + z_L\tanh(\gamma L)}\right]$$

$$= z_0\left[\frac{z_L + z_0\tanh(j\beta L)}{z_0 + z_L\tanh(j\beta L)}\right] \qquad [\text{Substituting } \gamma = j\beta]$$

$$\underbrace{=}_{\text{by } 10.38} z_0\left[\frac{z_L + jz_0\tan(\beta L)}{z_0 + jz_L\tan(\beta L)}\right]$$

SUMMARY

There are many identities showing relationships between hyperbolic and trigonometric functions. We can use these to evaluate trigonometric and hyperbolic functions of complex numbers.

5.27 $\dfrac{\sinh(x)}{\cosh(x)} = \tanh(x)$ 10.38 $\tanh(jz) = j\tan(z)$

Exercise **10(f)**

Solutions are given at the end of this additional material. Complete solutions are in this website.

1 Evaluate the following:

a $\cos(j)$

b $\sin(j)$

c $\tan(j)$

d $\sin(j\pi)$

2 Evaluate the following:

a $\cosh(j\pi)$

b $\sinh(j\ln(3))$

c $\tanh(j\pi/3)$

3 Show that

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

4 Show that

$$\tan(z) = \frac{1}{j}\left[\frac{e^{jz} - e^{-jz}}{e^{jz} + e^{-jz}}\right]$$

5 By using 10.27 and 10.28 show that

$$\cos^2(z) + \sin^2(z) = 1$$

6 Show that

$$\cos(jz) = \cosh(z)$$

7 Show that

$$\cosh(jz) = \cos(z)$$

8 Determine values of x and y for each of the following (x and y are real):

a $x + jy = \cos(1 + j\pi)$

b $x + jy = \sin(-1 - j\pi)$

c $x + jy = \tanh\left(-j\frac{\pi}{4}\right)$

9 [*electrical principles*] A cable has a voltage v at a distance x from the sending end, given by

$$v = V_L\left(\cosh(\gamma x) + \frac{z_0}{z_L}\sinh(\gamma x)\right)$$

where V_L is the load voltage, z_0 is the characteristic impedance, z_L is the load impedance and γ is the propagation coefficient. Show that if $\gamma = j\beta$ then

$$v = V_L\left(\cos(\beta x) + j\frac{z_0}{z_L}\sin(\beta x)\right)$$

10 [*electrical principles*] If a voltage v at a distance x along a cable is given by

$$v = I_L z_0 \sinh[(\gamma + j\beta)x]$$

show that

$$\frac{v}{I_L z_0} = \sinh(\gamma x)\cos(\beta x) + j\cosh(\gamma x)\sin(\beta x)$$

11 [*electrical principles*] The impedance, z_x, at a distance x along a transmission line is given by

$$z_x = z_0\frac{(z_0 + z_L)e^{\gamma x} + (z_L - z_0)e^{-\gamma x}}{(z_0 + z_L)e^{\gamma x} + (z_0 - z_L)e^{-\gamma x}}$$

where z_L is the load impedance, z_0 is the characteristic impedance and γ is the propagation coefficient. Show that

$$z_x = z_0\left[\frac{z_0\sinh(\gamma x) + z_L\cosh(\gamma x)}{z_0\cosh(\gamma x) + z_L\sinh(\gamma x)}\right]$$

Miscellaneous exercise **10 (extra)**

Solutions are given at the end of this additional material. Complete solutions are in this website.

21 Evaluate the following:

a $\cos(\ln(1) + j)$ **b** $\sin\left(\frac{\pi}{2} + j\right)$

22 [*electrical principles*]

i The current, I_x, in a transmission line at a distance x from the receiving end is given by

$$I_x = \frac{I_L}{2z_0}\left(e^{\gamma x}(z_0 + z_L) - (z_0 - z_L)e^{-\gamma x}\right)$$

where I_L is the load current, z_0 is the characteristic impedance, z_L is the load impedance and γ is the propagation coefficient. Show that

$$I_x = I_L\left(\sinh(\gamma x) + \frac{z_L}{z_0}\cosh(\gamma x)\right)$$

ii Evaluate I_x for $\gamma x = 0.01 + j0.1$, $z_L = 250\angle 10°\ \Omega$, $z_0 = 500\angle(-10°)\ \Omega$ and $I_L = 250$A.

23 [*control engineering*] The steady-state output, y_{ss}, of a stable system is given by

$$y_{ss} = C\left[\frac{e^{j\phi}e^{j\omega t} - e^{-j\phi}e^{-j\omega t}}{2j}\right]$$

where C is a real constant, ω = angular frequency, t = time and ϕ = phase. Show that

$$y_{ss} = C\sin(\omega t + \phi)$$

24 [*control engineering*] The following transformation is used to derive an equivalent digital filter from an analogue filter:

$$F(s) = \frac{s-1}{s+1}$$

where $s = e^{j\omega t}$ and T = sampling period. Show that

$$F(s) = j\tan\left(\frac{\omega T}{2}\right)$$

Solutions

2(g)

1 $i = i_1 + i_2 + i_3$

2 $i_1 + i_3 = i_2 + i_4 + i_5$

3 $i_1 + i_2 = 10$ mA, $i_1 = i_3 + i_4$, $i_3 + i_5 = 10$ mA and $i_2 + i_4 = i_5$

4 $i = 3$ mA

5 $i = 1$ mA

6 $i_1 = 5.67$ mA, $i_2 = 0.90$ mA

7 $i_1 = 4.06$ mA, $i_2 = 0.38$ mA, $i_3 = 0.11$ mA

8 $i_1 = 2.69$ mA, $i_2 = 1.92$ mA, $i_3 = 0.53$ mA

9 $i_1 = 5.90$ mA, $i_2 = 2.24$ mA, $i_3 = 1.17$ mA, $i_4 = 0.16$ mA

10 $i_1 = 3.64$ mA, $i_2 = 0.77$ mA, $i_3 = 0.24$ mA

3(g)

2 **a** $5H(t-7)$

b $H(t-5) - H(t-8)$

c $tH(t-1)$

d $2t\left[H(t-1) - H(t-2)\right]$

3 **i** $f(t) = t^3H(t)$, $g(t) = (t-2)^3 H(t-2)$

5(f)

1 0.266, 0.276, 1.000

2 1.862, −1.862, 0, 0.549, 1.812, 7.601, no real solution

3 π, 5, π, 0.236

4 **a** 1.12 **b** 1.86 **c** 0.55

10 240.87 m

11 $s = \dfrac{T}{w}\left[\sinh\left(\dfrac{wx}{T}\right) - \tan^{-1}(\theta)\right]$

ME5 (extra)

18 **b** −60 m

10(f)

1 **a** 1.54 **b** $j\,1.17$ **c** $j\,0.76$ **d** $j\,11.55$

2 **a** −1 **b** $j\,0.89$ **c** $j\sqrt{3}$

8 **a** $x = 6.26$, $y = -9.72$ **b** $x = -9.75$, $y = -6.24$ **c** $x = 0$, $y = -1$

ME10 (extra)

21 **a** 1.543 **b** 1.543

22 **ii** $119.325 + j\,67.618$